S1574: Relativity Without Light

Supplementary Notes

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1 The invariant expression for h'

Starting from the linear equations' solution, take a natural logarithm:

$$\begin{aligned} r &= \frac{(s_1 - w)(s_2 + v)}{(s_2 + w)(s_1 - v)} \\ \ln r &= \ln(s_1 - w) + \ln(s_2 + v) - \ln(s_2 + w) - \ln(s_1 - v) \end{aligned}$$

We want the partial derivative of this by v at 0. As $v \to 0$, $\frac{\partial \ln r}{\partial v} \to 0$. In addition:

$$\begin{array}{rclcrcl} \frac{\partial s_1}{\partial v} & \to & \frac{1}{h'(s)} & \quad & s_1 & \to & s \\ \frac{\partial s_2}{\partial v} & \to & -\frac{1}{h'(s)} & & s_2 & \to & s \\ \frac{\partial w}{\partial v} & \to & \frac{1}{h'(u)} & & w & \to & u \end{array}$$

Using the chain rule for natural logarithms, $\frac{\partial(\ln f)}{\partial v} = \frac{1}{f}\frac{\partial f}{\partial v}$, we evaluate and simplify:

$$\begin{aligned} 0 &= \frac{\frac{1}{h'(s)} - \frac{1}{h'(u)}}{s-u} + \frac{-\frac{1}{h'(s)} + 1}{s} - \frac{-\frac{1}{h'(s)} + \frac{1}{h'(u)}}{s+u} - \frac{\frac{1}{h'(s)} - 1}{s} \\ &= \left(\frac{1}{h'(s)} - \frac{1}{h'(u)}\right)\left(\frac{1}{s-u} + \frac{1}{s+u}\right) - \left(\frac{1}{h'(s)} - 1\right)\frac{2}{s} \\ &= \left[\left(\frac{1}{h'(s)} - 1\right) - \left(\frac{1}{h'(u)} - 1\right)\right]\frac{2s}{s^2 - u^2} - \left(\frac{1}{h'(s)} - 1\right)\frac{2}{s} \\ &= \left(\frac{1}{h'(s)} - 1\right)\left(\frac{2s}{s^2 - u^2} - \frac{2}{s}\right) - \left(\frac{1}{h'(u)} - 1\right)\frac{2s}{s^2 - u^2} \\ &= \left(\frac{1}{h'(s)} - 1\right)\frac{2u^2}{s(s^2 - u^2)} - \left(\frac{1}{h'(u)} - 1\right)\frac{2s}{s^2 - u^2} \end{aligned}$$

As long as the appropriate expressions aren't zero (s, u, and $s^2 - u^2$ among them), we can rewrite this as:

$$\left(\frac{1}{h'(s)} - 1\right)\frac{1}{s^2} = \left(\frac{1}{h'(u)} - 1\right)\frac{1}{u^2}$$

2 From h to a velocity addition law

We constructed h so that, if w is the velocity sum of u and v:

$$h(w) = h(u) + h(v)$$

From the last step we reached in class:

$$h(v) = \frac{1}{\sqrt{K}} \ln \sqrt{\frac{1+\sqrt{K}v}{1-\sqrt{K}v}}$$

Make the substitution and simplify:

$$\begin{aligned}
\frac{1}{\sqrt{K}} \ln \sqrt{\frac{1+\sqrt{K}w}{1-\sqrt{K}w}} &= \frac{1}{\sqrt{K}} \ln \sqrt{\frac{1+\sqrt{K}u}{1-\sqrt{K}u}} + \frac{1}{\sqrt{K}} \ln \sqrt{\frac{1+\sqrt{K}v}{1-\sqrt{K}v}} \\
\frac{1+\sqrt{K}w}{1-\sqrt{K}w} &= \frac{1+\sqrt{K}u}{1-\sqrt{K}u}\frac{1+\sqrt{K}v}{1-\sqrt{K}v} \\
&= \frac{1+Kuv+\sqrt{K}(u+v)}{1+Kuv-\sqrt{K}(u+v)}
\end{aligned}$$

Now we can divide the top and bottom of the right side by $1 + Kuv$:

$$\frac{1+\sqrt{K}w}{1-\sqrt{K}w} = \frac{1+\sqrt{K}\frac{u+v}{1+Kuv}}{1-\sqrt{K}\frac{u+v}{1+Kuv}}$$

From this we must have $w = \dfrac{u+v}{1+Kuv}$.