1 Introduction

Question 1. You want to color the sides of an equilateral triangle one of three colors: red, green, or blue. Two colorings are considered the same if you can rotate or reflect one triangle to get the other one. How many different colorings are there?

We saw that there are six different moves that change one coloring into an equivalent one. These are:

- e – do nothing
- r – rotate 120° right
- l – rotate 120° left
- f_1 – reflect across the line through the top vertex
- f_2 – reflect across the line through the right vertex
- f_3 – reflect across the line through the left vertex

Definition 2. The ***orbit*** of a coloring is all of the colorings that it is equivalent to.

Definition 3. The ***stabilizer*** of a coloring is all of the moves that leave in unchanged.

Definition 4. The ***fixed set*** of a move is all of the colorings that it does nothing to.

Example 5.

- The orbit of a triangle with only one color is of size 1 (just that triangle) and the stabilizer is of size 6 (all of the moves).
- The orbit of a triangle with two colors is of size 3 and the stabilizer is of size 2.
- The orbit of a triangle with three colors is of size 6 (all the triangles with all three colors) and the stabilizer is of size 1 (just the identity).

Example 6.

- The fixed set of the identity is of size 27 (all of the colorings)
- The fixed set of a reflection is of size 9 (the two sides next to the vertex you're reflecting across have to be the same color)
- The fixed set of a rotation is of size 3 (all three sides of the triangle have to be the same color)

2 Orbit-Stabilizer Theorem

Theorem 7 (Orbit-Stabilizer). *Consider any set of moves acting on any set of colorings. For any coloring*

$$\textit{size of orbit} \cdot \textit{size of stabilizer} = \#\textit{ of moves}$$

Example 8. Say the set of moves is the rotational symmetries of a cube. Let the set of colorings be those where the faces of the cube are all colored white except for one face colored black.

There are six colorings total and the orbit of any coloring is all six colorings. In addition, there are four moves that fix a coloring: do nothing and rotate 90°, 180°, or 270° through the face.

So all colorings have orbit of size 6 and stabilizer of size 4.

$$6 \times 4 = 24$$

Example 9. We let the set of moves be the rotational symmetries of a cube. This time, the set of colorings are those where we color the vertices of the cube all white except for one vertex that is colored black.

There are 8 colorings total and again the orbit of any coloring is all eight colorings. Here, the stabilizer of any element is of size 3: do nothing, and rotate by 120° or 240° through a space diagonal.

$$8 \times 3 = 24$$

Example 10. The set of moves is still the rotational symmetries of a cube. We let the set of colorings be those where all the edges of a cube are colored white except for one edge that is colored black.

There are 12 colorings total and each orbit is of size twelve again. This time the stabilizer of any coloring is only of size 2: do nothing, and rotate around an edge (you can think of this move as taking the rectangle made of two opposite edges and two face diagonals and rotating it 180°).

$$12 \times 2 = 24$$

For the orbit-stabilizer theorem to be true, it would have to be that there are 24 moves total. We can count that we have in fact found all of these moves:

- Rotating 90° in either direction through a pair of faces: **6** total
- Rotating 180° through a pair of faces: **3** total
- Do nothing: **1** total
- Rotating 120° in either direction through a space diagonal: **8** total
- Rotating a pair of edges 180°: **6** total

These add up to 24 moves total, which confirms what we thought the orbit-stabilizer theorem should say. Now we're going to prove the orbit-stabilizer theorem.

Proof. We're given one coloring. We want to take the size of its orbit and the size of its stabilizer and show that the product of these numbers is the number of moves we have.

We make a table which has one column for each of the colorings in the orbit. We place each of the moves in one of the columns based on where it takes our coloring. So the column above our coloring has "size of stabilizer" moves in it. What we want to show is that all the columns have the same height, so the number of things in the table total is "size of stabilizer" · "size of orbit". We already know that the number of things in the table is just "# of moves", so if this were true, we would know that the orbit-stabilizer theorem is true.

First we show that all of the columns are at least as tall as the first one. Say our coloring is called C and we're looking at the column belonging to coloring D. We already have some move (let's call it m) that takes C to D. We can make one move that takes C to D for each move that takes C to itself. Say we have a move called x that takes C to C. Then the move "do x then m" takes C to D.

This creates one move in D's column for each move in C's column, so we would be done with the first part if we knew that all the moves we just wrote down were different. However, if there were two different moves x and y where "do x then m" and "do y then m" were the same, then it would also be the case that the moves "do x then m then m backwards" and "do y then m then m backwards" would have to be the same move. But the first move is just x and the second is just y, which we said were different, so all of these moves in the form "do x then m" must all be different.

Now we just need to check that none of the columns are taller than the first column. This part is almost exactly the same as the first part. We have some coloring D and a bunch of moves that take C to D. Now we want to come up with a bunch of moves that take C to itself, one for each of the moves that take C to D.

Pick one of the moves that takes C to D. Let's call it m. Now for any move x that takes C to D, the move "do x then m backwards" takes C to itself. All of the moves in this form are the same for exactly the same reason. If "do x then m backwards" is the same as "do y then m backwards", then it would also be true that "do x then m backwards then m again" is the same as "do y then m backwards then m again", and these two are just x and y.

So all of the columns in this table must be the same height, so the number of elements in this table (# number of moves), must be its height (size of stabilizer) times its width (size of orbit), which is exactly what we wanted to show. □

3 Burnside's Lemma

The problem that we started with and the problem that we're working with in general is to count the number of *non-equivalent* colorings; that is, how many colorings are there that belong to different orbits. But this is just counting how many orbits there are total. Our answer just is:

$$\#\text{ of orbits}$$

Now to add up the number of orbits total, we assign the number $\frac{1}{\text{size of orbit}}$ to each of our colorings. When we add all of these numbers up, each orbit consists of "size of orbit" colorings each assigned the number $\frac{1}{\text{size of orbit}}$, which adds up to 1 for each orbit. Therefore, if we add up all of the numbers we assigned to all of the orbits we just get the number of orbits total. So our answer is:

$$\textbf{add up the number } \frac{1}{\text{size of orbit}} \textbf{ for each of the colorings}$$

Now we can use what we worked so hard to prove earlier. By the orbit-stabilizer theorem, we know that $\frac{1}{\text{size of orbit}} = \frac{\text{size of stabilizer}}{\text{size of orbit} \cdot \text{size of stabilizer}} = \frac{\text{size of stabilizer}}{\#\text{ of moves}}$. So the answer we had before is the same as saying:

$$\textbf{add up the number } \frac{\text{size of stabilizer}}{\#\text{ of moves}} \textbf{ for each of the colorings}$$

But "# of moves" is a fixed number, so dividing all the numbers by this and then adding them is exactly the same as adding up "size of stabilizer" for all the colorings and then dividing by "# of moves" at the end. Thus we can rewrite our answer as:

add up the number size of stabilizer **for each of the colorings**
then divide the total by # of moves

Finally, this is the tricky part. What we're doing above is adding up the sizes of the stabilizer of all the colorings. This just counts the total number of moves that leave each coloring unchanged. But if you want to count this, you can also think of it as counting the total number of colorings left unchanged by each of the moves. The "colorings left unchanged" by a moves is exactly what we called the fixed set of the move, so we can rewrite the answer to our problem as:

add up the number size of fixed set **for each of the moves**
then divide the total by # of moves

Which is exactly the same as saying:

the average size of all the fixed sets

So what we starting off trying to calculate, the number of orbits, turns out to be exactly equal to the average size of the fixed set of all the moves. This is called Burnside's Lemma.

The reason that Burnside's Lemma useful is that unlike all the other intermediate terms we had in this proof, it's fairly straightforward to calculate the size of all the fixed sets and average them. Below are two examples of how to use Burnside's Lemma.

4 Two Examples

4.1 Example 1: Triangles, Again

We're going to compute the number of orbits of the triangle problem that we started off with as an example. We have six moves: 2 rotations, 3 reflections, and the identity:

- For a coloring to be fixed by a rotation, it has to have all three of its sides the same color, so there are only 3 colorings in the fixed set of a rotation.
- A reflection switches two sides and leaves the last one unchanged, so there are 3 choices for the coloring of the pair of sides and 3 more choices for the coloring of the last side, for a total of 9 colorings in the fixed set of a reflection.
- Finally, every coloring is fixed by the identity, so all 27 colorings are in the fixed set of the identity.

Therefore, by Burnside's Lemma, there are a total of $\frac{3+3+9+9+9+27}{6} = 10$ orbits, exactly as we expected.

4.2 Example 2: Rotations on a Cube

This example is move complicated than the previous one, but it shows the power of Burnside's Lemma, since this a problem that is much harder to solve by hand.

Say we're trying to count the number of distinct colorings of the faces of a cube with the colors red, green, and blue. We say two colorings are the same if you can rotate the two cubes to get from one to another.

We already know what the moves are from Examples 8, 9, and 10. There are 24 of them and they're all listed on page 2. Now we just need to calculate the size of the fixed set of all of these moves.

- Rotating 90° through a face keeps two of the faces in the same place, but moves the other four around the equator. Thus the top and bottom face can be any of the 3 colors while the equatorial faces all have to be the same color and there are 3 more choices for this color, for a total of $3^3 = 27$ colorings in the fixed

- Rotating 180° through a face keeps two of the faces in the same place, but swaps the other two pairs of faces. There are four choices to be made here: the color of the top and bottom face, and the colors of the two opposite pairs of faces. This fixed set is of size $3^4 = 81$.
- Doing nothing fixes all $3^6 = 729$ colorings.
- Rotating 120° through a space diagonal moves the faces in two loops of 3 (the close faces and the far faces). A coloring fixed by this move has 3 choices for the color of the three closer faces and 3 choices for the color of the three further faces, for $3 \times 3 = 9$ total colorings.
- Rotating a pair of edges 180° is the hardest one to thing about, but what it does is swap three pairs of faces. Thus to make a fixed coloring, there are three choices: what color to make each of the pairs of faces. Thus there are $3^3 = 27$ colorings here.

This is pretty hard to visualize so it might make help if you take a physical cube and rotate it yourself. In the end, the number of distinct colorings is just the number of orbits, which by Burnside's Lemma is just the average size of the fixed set:

$$\frac{27 \cdot 6 + 81 \cdot 3 + 729 \cdot 1 + 9 \cdot 8 + 27 \cdot 6}{24} = 57$$

See if you can figure out how to solve the third problem from here. That is, we're done almost all the work for figuring out the number of distinct colorings for a cube with any number of paint, so see if you can figure out the number of distinct colorings of a cube with 1000 colors of paint. [The answer should be 41,666,792,167,000,000.]