Problems on Complex Number in Plane Geometry

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Problem 1 Given triangle ABC and a point M. D is on BC and $MD \perp MA$. Similarly, we get E and F. Prove that D, E, F are collinear.

Problem 2 Let $A_1A_2A_3A_4$ be a cyclic quadrilateral. H_i is the orthocenter of $\triangle A_{i+1}A_{i+2}A_{i+3}$, $i = 1, 2, 3, 4$ where $A_{k+4} = A_k$. Prove that H_1, H_2, H_3, H_4 are concyclic. Do the same thing to their centers of gravity and incenters.

Problem 3 Let $ABCD$ be a cyclic quadrilateral. P is an arbitrary point. E, F, G, H, I, J are the feet of perpendicular from P to AB, CD, AD, BC, AC, BD. L, M, N are midpoints of EF, GH, IJ. Prove that L, M, N are collinear.

Problem 4 Given a rhombus $ABCD$ and its incircle ω with the center O. E, F, G, H lies on AB, BC, CD, DA respectively and EH, FG are tangent to ω. Prove that EF is parallel to HG.

Problem 5 PA and PB are two tangent lines about a given circle ω whose center is O. A and B are tangent points. M is the midpoint of BP and D is the midpoint of OP. AM intersects ω at C. Line CD intersects ω at E. Prove that AE is parallel to OP.

Problem 6 ABC is a given triangle and P is a point interior to it. Triangle DEF is the pedal triangle with respect to P and ABC ($PD \perp BC$ at D). Suppose that $OP = d$, where O is the circumcenter of ABC and R is the circumradius of ABC. Prove that the area of triangle DEF, $S_{\Delta DEF} = \frac{1}{4}(1 - \frac{d^2}{R^2})S_{\Delta ABC}$.

Problem 7 $\triangle ABC$ and $\triangle DEF$ are congruent to each other in the opposite direction. Prove that midpoints of AD, BE, CF are collinear.

Problem 8 $A_0A_1 \ldots A_{11}$ is a regular 12-sided polygon. Prove that $|A_0A_1| + |A_0A_3| = 2r$, where r is the distance from the center to a side of the polygon.

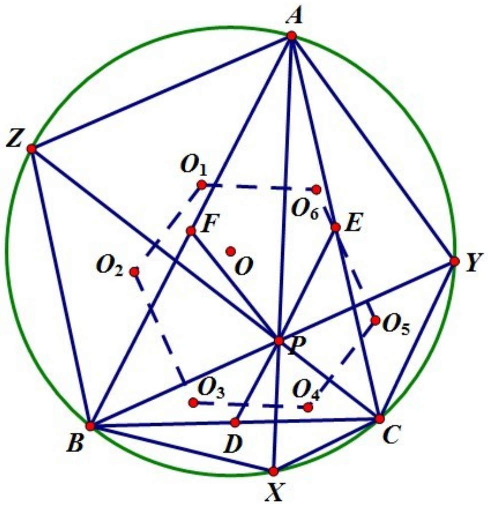
Problem 9 $A_0A_1 \ldots A_{14}$ is a regular 15-sided polygon.

Prove that (1) $A_0A_1 = A_0A_6 - A_0A_4$. (2) $\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_4} + \frac{1}{A_0A_7}$.

Problem * Let ABC be a triangle (not equilateral) with circumcircle ω. D, E and F are midpoints of BC, AC and AB, respectively. P is a point interior to triangle ABC such that the lengths of segments satisfy the equation $PD : PE : PF = BC : AC : AB$.

(1) Prove that P lies on the Euler line of ABC.

(2) AP, BP, CP intersect ω at X, Y, Z. $O_1, O_2, O_3, O_4, O_5, O_6$ are circumcenters of triangles $PAZ, PBZ, PBX, PCX, PCY, PAY$ respectively. Prove that those six circumcenters are concyclic.



Problem * Given an inequilateral triangle ABC and a straight line l. l_1, l_2, l_3 pass through A, B, C respectively and are parallel to l. Let l_1' be the reflection of l_1 with respect to BC. Define l_2' and l_3' similarly. Prove that l_1', l_2', l_3' are concurrent if and only if l is parallel to the Euler line of triangle ABC.