# Vectors and Beyond: Day 1 Notes

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## 1 Notes

### 1.1 Vector Spaces

We defined a *vector space* as a collection of objects we'd call *vectors*, for which we had two operations: addition and scalar multiplication.

Addition combines two vectors to give another vector and is subject to the following rules for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$:

- (Associativity) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- (Commutativity) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- The existence of an additive identity $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$
- The existence of a $-\mathbf{v}$ for every $\mathbf{v}$ such that $(-\mathbf{v}) + \mathbf{v} = \mathbf{0}$

Scalar multiplication combines a vector and an ordinary number (a *scalar*) to give another vector. It is subject to the following rules for all vectors $\mathbf{u}$ and $\mathbf{v}$, and all scalars $a$ and $b$:

- (Distributive law)

$$\begin{aligned} a(\mathbf{u} + \mathbf{v}) &= a\mathbf{u} + a\mathbf{v} \\ (a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v} \end{aligned}$$

- $(ab)\mathbf{v} = a(b\mathbf{v})$
- The existence of a multiplicative identity 1 such that $1\mathbf{v} = \mathbf{v}$

The rules that the scalars must follow are those of a *field*. Essentially, a field is a collection of numbers where you can add, multiply, take negatives, and take reciprocals of everything that's not zero. Examples include the collection $\mathbb{R}$ of all real numbers, the collection $\mathbb{C}$ of all complex numbers, and the collection $\mathbb{Q}$ of all rational numbers. The integers ($\mathbb{Z}$) are *not* an example, since the reciprocals of integers aren't always integers! You can see http://en.wikipedia.org/wiki/Field_(mathematics) for a precise explanation.

Notice that this definition says nothing about the dimension of the vector space! Nor does it say anything about dot products; a vector space with a dot product is known as an *inner product space*.

### 1.2 Dot Products

The dot product takes two vectors and returns a scalar. The answer is related to the lengths of the vectors and how close they are to being in the same direction.

If $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ are two vectors, then the formula for their dot product is:

$$\mathbf{u} \cdot \mathbf{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Given a dot product on a vector space, we can define a vector's magnitude with:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

### 1.3 Cross Products

The cross product (which only works in a 3-dimensional space) takes two vectors and returns another vector. The answer is related to the lengths of the two vectors and how close they are to being perpendicular.

If $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ are two vectors, then the formula for their cross product is:

$$\mathbf{u} \cdot \mathbf{v} = (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1)$$

## 2 Problems

1. Using the Law of Cosines, prove in two dimensions that:

   $$|\mathbf{u} \cdot \mathbf{v}| = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos \theta$$

   ...where $\theta$ is the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$.

2. (difficult) Prove the Cauchy-Schwarz Inequality: that for vectors $\mathbf{u}$ and $\mathbf{v}$ (in a space of any dimension):

   $$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \, ||\mathbf{v}||$$

3. (easier) Prove, using the result of problem 1, that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (have an angle of $\pm 90°$) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

## 3 Problem Hints

- Problem 3 is the easiest; you should probably start with that one.
- For problem 2, try playing with $||\mathbf{u} + k\mathbf{v}||$.