Numbers That Do Weird Things, Week 1

HSSP Summer 2016

2016-07-10

1. Trigonometric Representation of Complex Numbers and DeMoivere's Theorem

   (a) There's a really nice way to represent complex numbers using trigonometric functions. Show that any comblex number $a + bi$ can be written in the form $k(\cos\theta + i\sin\theta)$, where k is real and $\theta \leq \theta < 2\pi$.

   (b) De Moivre's Theroem says that for any integer n,

   $$(\cos x + i\sin x)^n = \cos n + i\sin nx$$

   . Prove this. (Hint: use induction for positive integers, then tackle the negative integers! If you haven't heard of induction, follow a and b below.)

      i. Show that the theorem holds for $n = 0$ and $n = 1$.

      ii. Suppose that the theorem holds for $n = k$; that is,

      $$(\cos x + i\sin x)^k = \cos kx + i\sin kx$$

      for some k. Show that it holds for $n = k + 1$; that is,

      $$(\cos x + i\sin x)^{k+1} = \cos((k+1)x) + i\sin((k+1)x)$$

      . This, combined with part (i), allows us to conclude that the theorem is true for all nonnegative integers n. Convince yourself of this!

      iii. Using part (b), prove the theorem for negative integers.

2. 2D Rotation Matrix

   In class, we talked about how complex numbers can be seen as rotations and how they can be represented as a matrix. Using Part (a) of Problem 1, find the 2D rotation matrix that will rotate a poit about the origin by an angle of θ. Then, use this to prove that $\sin\alpha + \beta = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

3. Subalgebras

   In class, we talked about how the real numbers, $\mathbb{R}$, are an algebra. A subalgebra of an algebra A is a subset of A that is itself an algebra. What are the subalgebras of $\mathbb{R}$?

4. Matrix Representation of Quaternions

   In class, we talked about how the complex numbers can be represented as 2x2 matrices of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, where a and b are real. We can also write the quaternions like this!

   (a) Find 2x2 matrices for i, j, and k that satisfy the conditions we had for the quaternions ($i^2 = j^2 = k^2 = ijk = -1$).

   (b) Show that, using real numbers as scalars, you can write any quaternion as a matrix of the form $aE + bI + cJ + dK$ where E is the identity matrix, I is your matrix for i, J is your matrix for j, and K is your matrix for k.

   (c) Does every 2x2 matrix correspond to a quaternion? Does every matrix that corresponds to a quaternion correspond to a unique quaternion? For each of these questions: If so, try to prove it. If not, find a counterexample.

5. (1985 AIME Problem 3) Find c if a, b, and c are positive integers which satisfy $c = (a+bi)^3 - 107i$.

6. (2009 AIME 1 Problem 2) There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

   Find n.

7. (1999 AIME Problem 9) A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers, find $m + n$.