About these notes

Warning! The notes you have in your hands are not a textbook, but an outline for a Moore method course on point-set topology. It contains definitions and theorems, exercises and questions, along with a few remarks that provide motivation and context. However, there are no proofs and few explanations your job is to discover these for yourself. Even more so than with the average math textbook, you cannot simply read this text! Rather, treat it as a set of guide posts to help you find your own way through a rugged but beautiful terrain. You are not the first to visit this magnificent landscape, but you can still experience the thrill of exploring it for yourself. Its like hiking up to a mountain summit instead of taking the tour bus.

You are not setting out on this trek alone: you have your classmates to help you. In class, students will take turns presenting proofs of statements and critiquing each others arguments. What one person is confused about, the class as a whole should be able to unravel (with only minimal participation by me). For this system to work, it is crucial that you treat your fellow students with sensitivity and respect but dont let them get away with a bad argument! And when you are the presenter yourself, do not be afraid of getting stuck or of making a mistake; its no big deal. It happens to everyone, and it is the only way to make progress in mathematics.

In $\flat$-statements, the proofs are relatively routine. They may be tedious sometimes, so it may not be worth your while to write out the entire proof in detail. (However, if you are not sure how the proof would go, you are encouraged to do some part of it, or to try a few examples of your own devising, to help you understand the statement; of course, this is always a good idea!) Statements labelled "Challenge" are particularly difficult. Statements labelled "Fuzzy" are vaguely-phrased questions that hint at an interesting notion or result; they are an opportunity for you to unleash your creativity and come up with your own conjectures.

We're going to start working through the problems, and proceed at whatever pace you work at. I have no idea how far we'll get, but hopefully we'll at least get to some more interesting parts. I can't force you to work outside of class, and won't try; however, if you spend time working on your own, we'll be able to go faster and you'll learn more and probably have more fun. Hopefully the class is exciting enough that you feel compelled to think about topology outside of class. And after HSSP is over, you can still work on the problems here! I'll be happy to give you tips and answer any questions; just email me.

Disclaimers: These notes are (almost) entirely taken from notes for a similar class Alfonso Gracia-Saz taught at Canada/USA MathCamp 2014, which are available at `http://www.math.utsc.utoronto.ca/c27/alfonso.pdf`.

These notes are incomplete. I will continue to add to them over the course of HSSP, making sure to keep ahead of where the class is.

1 The definition of topology

> We are about to introduce the main objects of this course: topologies. The definition may appear at first random and capricious, and you may wonder why we should care about it. Later on, we will motivate where this definition comes from and why it is a useful one. For now, we want to concentrate simply on getting familiar with the concept.

Definition 1.1. Let X be a set a *topology* on X is a family τ of subsets of X which satisfies three properties (spelled out below). We will say that a subset of X is an open set iff it is an element of τ. The three properties are:

(T1) The total set and the empty set are open sets.

Equivalently, $X \in \tau$ and $\emptyset \in \tau$.

(T2) The intersection of any two open sets is an open set.

Equivalently, if $A, B \in \tau$, then $A \cap B \in \tau$.

(T3) The union of open sets (no matter how many, including infinitely many) is an open set.

Equivalently, if I is a set of indices and $A_i \in \tau$ for all $i \in I$ then $\bigcup_{i \in I} A_i \in \tau$.

Or, equivalently, for any $\sigma \subseteq \tau$, $\bigcup_{B \in \sigma} B \in \tau$.

A topological space is a pair (X, τ) where X is a set and τ is a topology on X.

Exercise 1.2. ($\flat$) Among the following, some are topologies on the set $\mathbb{Z}$ and some are not. Which ones are? If an example is not a topology, but you can modify it slightly to make it into a topology, do so. If an example is a topology and you can generalize it into more examples, do so.

(a) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V\}$. In words, a set is open iff it contains 0.

(b) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \notin V\}$. In words, a set is open iff it does not contain 0.

(c) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V \text{ and } 1 \in V\}$.

(d) $\tau = \{V \subseteq \mathbb{Z} \mid 0 \in V \text{ or } 1 \in V\}$.

(e) $\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is finite}\}$.

(f) $\tau = \{V \subseteq \mathbb{Z} \mid V \text{ is infinite}\}$.

Exercise 1.3. Among the following, which ones are topologies on the set $\mathbb{R}$ and which ones are not?

(a) $\tau = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

(b) $\tau = \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

Exercise 1.4. Let X be any set.

(a) What is the topology on X that has the most open sets? This is called the *discrete* topology on X.

(b) What is the topology on X that has the least open sets? This is called the *indiscrete* topology on X.

Exercise 1.5. Let X be an arbitrary set. Which ones of the following are topologies?

(a) The *cofinite* topology: A set $V \subseteq X$ is open iff $[X \setminus V$ is finite or $V = \emptyset]$.

(b) The *coinfinite* topology: A set $V \subseteq X$ is open iff $[X \setminus V$ is infinite or $V = \emptyset$ or $V = X]$.

(c) The *cocountable* topology: A set $V \subseteq X$ is open iff $[X \setminus V$ is countable or $V = \emptyset]$.

Note: "Countable" includes "finite" as a particular case.

Exercise 1.6. Let (X, τ) be a topological space. Prove each of the following statements true or false.

(a) The intersection of any three open sets is open.

(b) The intersection of finitely many open sets is open.

(c) The intersection of open sets is open.

Definition 1.7. Let $x \in \mathbb{R}^N$ and let $\varepsilon > 0$. The *ball* centered at x with radius ε is

$$B_\varepsilon(x) := \{y \in \mathbb{R}^N \mid d(y, x) < \varepsilon\}$$

where $d(y, x)$ is the Euclidean distance between the points x and y.

Exercise 1.8. ($\flat$) Describe geometrically what a ball is in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

Definition 1.9. We define the *standard topology* or the *usual topology* on $\mathbb{R}^N$ as follows. Let $V \subseteq \mathbb{R}^N$. We say that V is *open* (in this topology) iff the following property is true: "For every $x \in V$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq V$." When we refer to a topology on $\mathbb{R}^N$ or to open sets on $\mathbb{R}^N$ without specifying which topology, we mean the standard one. This is the topology one often uses in analysis.

Exercise 1.10. Prove that the topology in Definition 1.9 is actually a topology.

Exercise 1.11. Show which ones of the following examples are open according to Definition 1.9:

(a) ($\flat$) The set $\{1\}$ in $\mathbb{R}$.

(b) ($\flat$) The interval $(2, 5)$ in $\mathbb{R}$.

(c) The ball $B_\delta(y)$ in $\mathbb{R}^N$ for any $y \in \mathbb{R}^N$ and any $\delta > 0$.

(d) ($\flat$) The interval $[0, 1)$ in $\mathbb{R}$.

(e) ($\flat$) The set $\{(x, y) \in \mathbb{R}^2 \mid x > y\}$ in $\mathbb{R}^2$.

Exercise 1.12. Find all the topologies on the set $X = \{0, 1, 2\}$.

Fuzzy 1.13. Look back at your answer to Exercise 1.12. Some of those topologies are very similar. One could even say that they are practically "the same topology" with different names. Come up with a definition of what practically the same topology could mean. Also, come up with a better name. With this definition, how many essentially different topologies are there on $\{0, 1, 2\}$?

2 Sequences and limits

In this chapter we will talk about limits and accumulation points of sequences in any topological space. Whenever we have a topology, we have a notion of limit, even if there is not a distance or a notion of "being close". Challenge 2.16 at the end of the chapter is the first surprise of the course and it illustrates how sequences do not quite behave in general the way you have gotten used to. This will be one *leitmotif* of this course.

Definition 2.1. Let X be a set. A *sequence* in X is a map $x : \mathbb{N} \to X$. Notice that in this course we will include 0 in $\mathbb{N}$. As notation, we often write x_n instead of $x(n)$ for an element in X. We may also write (x_n) or $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n=0}^{\infty}$ to refer to the whole sequence.

Definition 2.2. Let $P(n)$ be a statement the depends on a natural number $n \in \mathbb{N}$. We say that "$P(n)$ is eventually true for all n" if there exists $n_0 \in \mathbb{N}$ such that $P(n)$ is true for all $n \geq n_0$. If there is no ambiguity, we will say simple that "$P(n)$ is eventually true."

Definition 2.3. Let (X, τ) be a topological space. Let (x_n) be a sequence in X. Let $a \in X$. We say that a is a *limit* of the sequence when the following statement is true: "If $V \subseteq X$ is an open set such that $a \in V$, then $x_n \in V$ eventually for all n." In this case we say that the sequence *converges* to a. In words, this means that every open set containing a has to contain all the sequence, except for the first few terms.

We say that a sequence is *convergent* if it has at least one limit.

Exercise 2.4. Consider the sequence $0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \ldots$ on the set $X = \mathbb{R}$. For each of the following topologies, find all of its limits.

(a) the discrete topology,

(b) the indiscrete topology,

(c) the cofinite topology,

(d) ($\flat$) the cocountable topology,

(e) ($\flat$) the topology in Exercise 1.3,

(f) the standard topology.

Lemma 2.5. Let (X, τ) be a topological space. Let (x_n) be a sequence in X. Let $a \in X$. Prove that the following two statements are equivalent:

1. "For every open set $V \subseteq X$ such that $a \in V$ and for every $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ such that $x_n \in V$."
2. "For every open set $V \subseteq X$ such that $a \in V$, there are infinitely many $n \in \mathbb{N}$ such that $x_n \in V$." In words, every open set containing a contains infinitely many terms of the sequence.

Definition 2.6. In the situation of Lemma 2.5, when the two equivalent conditions are satisfied, we say that a is an *accumulation point of the sequence.*

Proposition 2.7. ($\flat$) Every limit of a sequence is also an accumulation point.

Exercise 2.8. Repeat Exercise 2.4, but this time find all the accumulation points instead of all the limits.

Fuzzy 2.9. Look back at Exercise 2.4. It included some examples where a sequence has more than one limit. Think of the discrete and indiscrete case; if a topology has more open sets, are sequences more or less likely to have multiple limits? Try to prove that in $\mathbb{R}$ with the usual topology, no sequence can have more than one limit. Which other topologies satisfy that? Can you come up with a necessary condition or a sufficient condition for a topology not to have sequences with multiple limits?

Exercise 2.10. Let C be the set of students who came to class today. We define a topology on C as follows. Given $V \subseteq C$, we say that V is open iff it satisfies the following property: "If $x \in V$ and y is sitting immediately to the left of x, then $y \in V$."

(a) Prove that this is actually a topology.

(b) Consider the sequence Aaron, Cindy, Aaron, Cindy, Aaron, Cindy, ... Find all its limits.

(c) Find all the accumulation points of the same sequence.

Exercise 2.11. Consider the set $\mathbb{Z}$ with the cofinite topology. Find an example of a sequence such that (or prove that such an example does not exist):

(a) it has more than one limit,

(b) it has exactly one limit and exactly one accumulation point,

(c) it has exactly one limit and it has more than one accumulation point,

(d) it has no limits and no accumulation points,

(e) it has no limits and it has exactly one accumulation point,

(f) it has no limits and it has more than one accumulation point.

Exercise 2.12. Describe all convergent sequences in $\mathbb{R}$ with the cocountable topology.

Definition 2.13. Let (X, τ) be a topological space. Let $x : \mathbb{N} \to X$ be a sequence. A *subsequence* of x is a sequence of the form $x \circ \lambda$ where $\lambda : \mathbb{N} \to \mathbb{N}$ is a strictly increasing map. As notation, if we write $x_n := x(n)$ for each $n \in \mathbb{N}$ and $n_k := \lambda(k)$ for each $k \in \mathbb{N}$. we will often write than $(x_{n_k})_k$ is a subsequence of $(x_n)_n$. If there is no danger of ambiguity, we may write simply that (x_{n_k}) is a subsequence of (x_n).

Proposition 2.14. Let (x_n) be a sequence in the topological space (X, τ) and let $a \in X$. Assume that a is a limit of a subsequence of (x_n). Prove that a is an accumulation point of (x_n).

Proposition 2.15. Let (x_n) be a sequence in $\mathbb{R}$ with the standard topology. Let $a \in X$. Assume that a as an accumulation point of (x_n). Prove that a is the limit of some subsequence of (x_n).

Challenge 2.16. Give an example that shows that Proposition 2.15 may fail if we use an arbitrary topological space.

3 Closed Sets

You know what an open set; now you are ready to learn what a closed set is. Just be careful and remember that closed does not equal "not open" (see Note 3.5). Closed sets give us an alternative way to think of a topology without mentioning open sets (see Theorem 3.8 and Note 3.9). Finally, you should look at results 2.14, 2.15, and 2.16 in parallel with 3.13, 3.14, and 3.15. Notice the pattern.

Definition 3.1. Let (X, τ) be a topological space. Let $A \subseteq X$. We say that A is *closed* when $X \setminus A$ is open.

Exercise 3.2. ($\flat$) Which ones of the following sets are closed in $\mathbb{R}$ with the standard topology?

- $A = [0, 1]$,
- $B = \{0\}$,
- $C = (0, 1]$,
- $D = [0, \infty)$,
- $E = \{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$,
- $F = E \cup \{0\}$.

Exercise 3.3. For each of the following topological spaces (X, τ), give examples of subsets $A_1, A_2, A_3, A_4 \subseteq X$ such that A_1 is open but not closed, A_2 is closed but not open, A_3 is both open and closed, and A_4 is neither open nor closed (or prove that such subsets do not exist). You may not use X or $\emptyset$ as any of your subsets

(a) X is the topological space of Exercise 2.10 (the topology on the students in the class).

(b) $X = \mathbb{Z}$ with the cofinite topology.

Definition 3.4. Let (X, τ) be a topological space. A set $A \subseteq X$ is called *clopen* if it is both open and closed, and called *ajar* if it is neither open nor closed.

Note 3.5. If Definition 3.4 makes you uncomfortable, you are not alone. Watch the following video: `http://youtu.be/SyD4p8_y8Kw`

Warning: The video contains profanity and dark humor. If you think such things may offend you, please do not watch the video.

Theorem 3.6. Let (X, τ) be a topological space. Let $\mathcal{F}$ denote the collection of closed sets. Then the following properties are true:

(C1) $X \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$.

In words, the total set and the empty set are closed sets.

(C2) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

In words, the union of any two closed sets is a closed set.

(C3) For any $\Lambda \subseteq \mathcal{F}$, $\bigcap_{B\in\Lambda} B \in \mathcal{F}$.

In words, the intersection of closed sets (no matter how many, including infinitely many) is a closed set.

Exercise 3.7. Show with an example in the standard topology that the arbitrary union of closed sets may not be closed.

Theorem 3.8. Let X be a set. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets that satisfies conditions (C1), (C2), and (C3) from Theorem 3.6. Then there exists a unique topology on X than has $\mathcal{F}$ as the family of closed sets.

Note 3.9. Theorem 3.8 is very important and it is easy to miss the deep implications it has. Until now, whenever we wanted to define a topology on a set, we would define the family of open sets. We could define the open sets to be any family we wanted, as long as they satisfied conditions (T1), (T2), (T3). Theorem 3.8 says that, instead, we may chose to define a topology by saying who the closed set are, and that we can choose any family of subsets to be the closed sets as long as they satisfy (C1), (C2), (C3). If they do, we do not need to worry about what the open sets are or about checking that they satisfy (T1), (T2), (T3). It will come for free. The following example shows that some topologies are more naturally defined by saying what the closed sets are than by saying what the open sets are.

Exercise 3.10. ($\flat$) Let X be a set. We are going to define two topologies on X. You already proved that they were topologies (back in Exercise 1.5). Show again that they are topologies, but this time using Theorem 3.8 and Note 3.9. Notice that the proofs are now shorter and more natural.

(a) The *cofinite* topology on X is the topology where the closed sets are the finite sets and X.

(b) The *cocountable* topology on X is the topology where the closed sets are the countable sets and X.

Definition 3.11. Let (X, τ) be a topological space. Let $A \subseteq X$. We say that A is *sequentially closed* when it satisfies the following property: "Let (x_n) be a sequence in X and let $a \in X$ be a limit of the sequence. Assume that $x_n \in A$ for all $n \in \mathbb{N}$. Then $a \in A$."

Exercise 3.12. Prove that $(0, 1]$ is not sequentially closed in $\mathbb{R}$ with the standard topology. Prove it directly from Definition 3.11, without using Proposition 3.14 below.

Proposition 3.13. Let (X, τ) be a topological space and let $A \subseteq X$. Prove that if A is closed, then A is sequentially closed.

Proposition 3.14. Consider $\mathbb{R}$ with the standard topology. Let $A \subseteq \mathbb{R}$. Prove that if A is sequentially closed, then A is closed.

Challenge 3.15. Give an example of a topological space (X, τ) and a subset $A \subseteq X$ such that A is sequentially closed but A is not closed.

4 Neighborhoods

There are a few things that are easier to do in the standard topology because it has balls. Topologies in general do not have balls, but they do have *neighborhoods*. To a certain extent, neighborhoods are like balls. Lets get acquainted.

Challenge 4.6 below is very important. If you read it next to Theorem 3.8 you will see one of the themes of this course: open sets are not the only way to define a topology.

Definition 4.1. Let (X, τ) be a topological space. Let $A \subseteq X$. We say that A is an *open neighborhood of* x when A is open and $x \in A$.

Definition 4.2. Let (X, τ) be a topological space. Let $x \in X$. A *basis of open neighborhoods of* x in (X, τ) is a family of open set $\mathcal{B}_x \subseteq \tau$ such that

- W is an open neighborhood of x for every $W \in \mathcal{B}_x$.
- If V is an open neighborhood of x, then there exists $W \in \mathcal{B}_x$ such that $W \subseteq V$.

Exercise 4.3. Let $x \in \mathbb{R}$. Among the following families of sets, which ones are bases of open neighborhoods of x in $\mathbb{R}$ with the standard topology?

For this exercise, do not write lengthy proofs. Just answer "yes" or "no" for each candidate, and, if needed, give a one-line explanation at most.

(a) $\{(x - \varepsilon, x + \varepsilon) \mid \varepsilon > 0\}$

(b) $\{(x - 1, x + \varepsilon) \mid \varepsilon > 0\}$

(c) $\{[x - \varepsilon, x + \varepsilon] \mid \varepsilon > 0\}$

(d) $\{(x - \varepsilon, x + 2\varepsilon) \mid \varepsilon > 0\}$

(e) $\{(x - \frac{1}{n}, x + \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$

(f) $\{(x - \frac{1}{n}, x + \frac{1}{n}) \mid n \in \mathbb{Z}^+, n > 100, n \text{ is odd}\}$

(g) $\{(x - \varepsilon, x + \varepsilon) \cup (x + 2\varepsilon, x + 3\varepsilon) \mid \varepsilon > 0\}$

(h) $\{(a, b) \mid a < x < b\}$

(i) $\{V \subseteq \mathbb{R} \mid V \text{ is open and } x \in V\}$

Exercise 4.4. For each of the following topological spaces, come up with a basis of open neighborhoods of each point which is as "small" as you can make it.

(a) A discrete topological space.

(b) An indiscrete topological space.

(c) The set of students in class with the topology of Exercise 2.10.

(d) The cofinite topology on $\mathbb{Z}$.

Exercise 4.5. Let (X, τ) be a topological space. Let $a \in X$ and let $\mathcal{B}_a$ be a basis of open neighborhoods of a in (X, τ). Write a condition equivalent to "a is a limit of the sequence (x_n) in the topological space (X, τ)" which uses the basis of open neighborhoods instead of open sets in general. Then prove they are equivalent.

Challenge 4.6. So far we know how to define a topology by saying who the open sets are, or by saying who the closed sets are. Find out a way to define a topology by saying who the neighborhoods of each point are. Specifically, list a set of axioms such that:

- If (X, τ) is a topological space, and $\mathcal{B}_x$ is a basis of open neighborhoods for x for every $x \in X$, then the families $\mathcal{B}_x$ satisfies the set of axioms.
- Let X be a set (not a topological space). Assume that for every $x \in X$ we have a family $\mathcal{B}_x \subseteq \mathcal{P}(X)$ and that they satisfy the axioms. Then there exists a unique topology on X which has $\mathcal{B}_x$ as a basis of neighborhoods of x for all $x \in X$.

Note: The existence part of the proof is tricky, and you may think you are done before you are. First, you need to define a topology. Then, you need to check it is a topology. Finally, you need to prove that this topology does have the original families $\mathcal{B}_x$ as bases of open neighborhoods.

Once we have this theorem, we may choose to describe a topology by giving a basis of open neighborhoods of each point in X instead of by describing the open sets. We are allowed to choose these bases any way we want as long as they satisfy the axioms. If you think about the definition of the standard topology in $\mathbb{R}^N$, you will notice that it makes more sense to define it in terms of open neighborhoods than in terms of open sets, just like it makes more sense to definite the cofinite topology in terms of closed sets than in terms of open sets.

Exercise 4.7. If you have taken a calculus or analysis class, you may have learned the definition of three "different" concepts. Given a sequence (x_n) in $\mathbb{R}$, you may have defined the concepts

$$\lim_{n\to\infty} x_n = a \in \mathbb{R}, \qquad \lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} x_n = -\infty.$$

These three concepts are not so different if we look at them from the lens of topology!

Consider the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. Define a topology on $\overline{\mathbb{R}}$ such that the three notions of limit are all particular cases of the topological definition of limit.

Exercise 4.8. We are going to prove the infinitude of primes using topology! Given $a, b \in \mathbb{Z}$, let us define the following set of integers: $S_{a,b} := \{a + nb \mid n \in \mathbb{Z}\}$.

(a) Prove that there exists a topology τ_p on $\mathbb{Z}$ that has $\{S_{a,b} \mid b \neq 0\}$ as a basis of open neighborhoods of a for every $a \in \mathbb{Z}$.

(b) Prove that for every $a, b \in \mathbb{Z}$ with $b \neq 0$, $S_{a,b}$ is clopen on $(\mathbb{Z}, \tau_p)$.

(c) Note that

$$\mathbb{Z} \setminus \{1, -1\} = \bigcup_{\text{primes } p} S_{0,p}$$

and that $\{1, -1\}$ is not open. Now, assume there are finitely many primes, and get a contradiction.