1 Generation of Symmetric Figures

Last time we have classified all the possible symmetries of finite plane figures. But how do we generate these nice symmetric figures to begin with? The general strategy is to start with a random (non-symmetric) pattern, make multiple copies of it, and apply the symmetry actions of the desired symmetry pattern on these copies. Then, the combined pattern will exhibit the desired symmetry pattern.

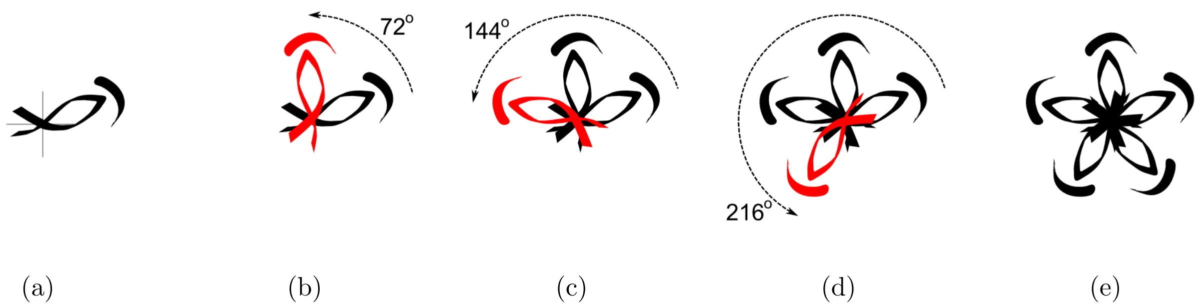


Figure 1: Generation of 5-fold rotation symmetric pattern.

For example, suppose we want to generate a pattern with 5-fold rotation, which are rotations by multiples of 72°. To do so, first produce some random shape as in Fig. 1(a). Then, make a copy of that shape and rotate it by 72°. After that, make yet another copy and rotate it by $2 \times 72° = 144°$. Continuing in this way, we will end up with a 5-fold pedal as in Fig. 1(e).

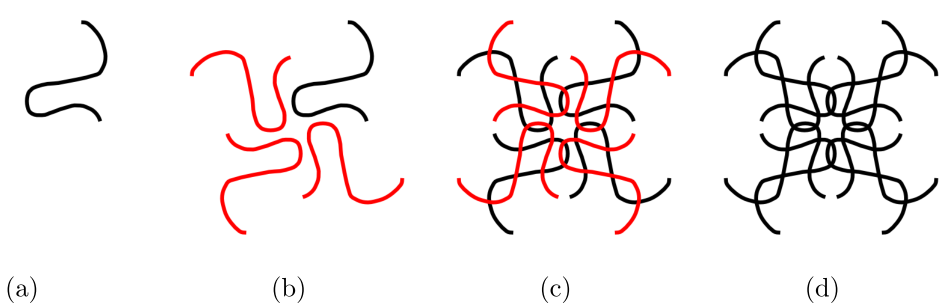


Figure 2: Generation of a 4-fold rotation and 4-fold reflection symmetry pattern.

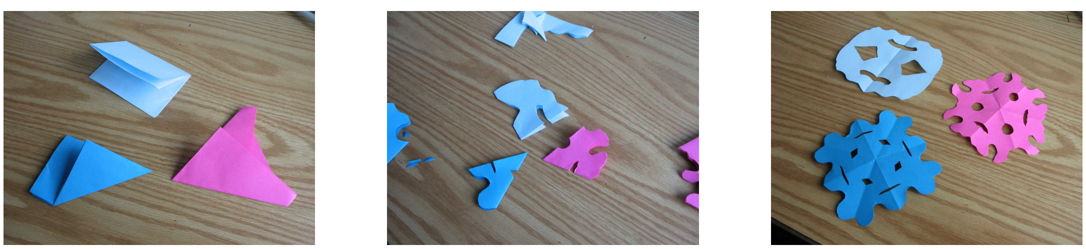
Similarly, suppose we want to produce a pattern with 4-fold rotation plus 4 distinct reflections. Start again with a random pattern (Fig. 2(a)). Make copies and rotate each by a different multiple of 90° to produce a 4-fold rotation symmetric pattern (Fig. 2(b)). After that, make a copy of that 4-fold rotation symmetric pattern and reflect it along the vertical line through the center (Fig. 2(c)). Together with the original copy, the combined figure now has a 4-fold rotation and 4 distinct reflections (Fig. 2(d)).

These descriptions are good on paper (or on computer[1]), but can we apply the idea hands-on? The answer is yes. Here are two ways.

Paper Cutting

1. Start with a piece of paper (need not be regular), make one or more folds. The fold lines should align on two straight edges (see Fig. 3(a)).
2. Cut away portions of the paper. Keep the portion that contains the sharp vertex where the fold lines meet (see Fig. 3(b)).
3. Unfold the paper (see Fig. 3(c)).

This procedure results in patterns that fall into the dihedral family. That is, it has n rotations and n distinct reflections.



(a) (b) (c)

Figure 3: Generation of a dihedral symmetry pattern via paper cutting.

[1] http://www.mathsisfun.com/geometry/symmetry-artist.html

Transparency on Foam Board Template

1. Start with a template that has a circle with n rays radiating from the center (Fig. 4(a)).

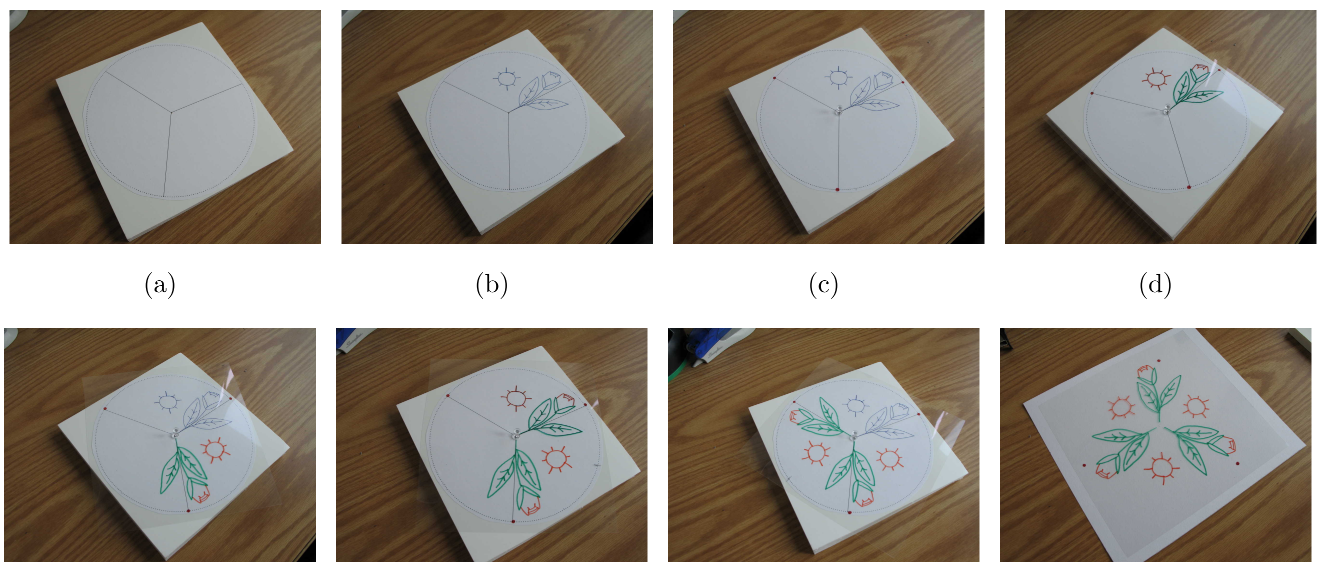
2. Make some random pattern on top of the template (Fig. 4(b)).

3. Use a push pin to pin a transparency on top of the template. The pin should be at the center of the circle. Also, mark the points where the rays meet the circle on the transparency (Fig. 4(c)).

4. Copy the pattern on the template onto the transparency (Fig. 4(d)).

5. Rotate the transparency clockwise until the marked points from step 3 align again (Fig. 4(e)).

6. Repeat step 4 and 5 until a symmetric pattern is produced (Fig. 4(f)–4(h)).



(a) (b) (c) (d)

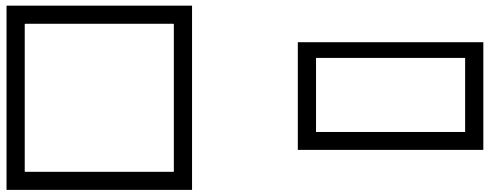
(e) (f) (g) (h)

Figure 4: Generation of a cyclic symmetry pattern via transparencies on foam template.

This procedure results in a pattern that falls into the cyclic family. That is, it has n rotations but no reflections.

2 Sub-Symmetry

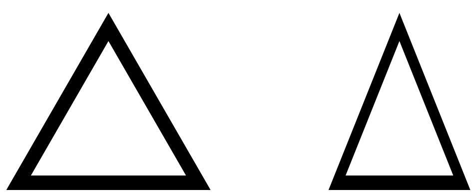
Last time we see that two finite plane figures may have the same symmetry or different symmetries. But sometimes two figures may look related to each other even if they don't share the same symmetry.



(a) A square (b) A rectangle

Figure 5: Square and rectangle.

A prime example would be the square and the rectangle (Fig. 5). A square can be thought of as a special rectangle in which the width and the height happens to be the same; Conversely, a rectangle can be considered as a distorted square. Intuitively, we would like to say that the square has a "higher" symmetry than the rectangle, but what do we mean by that, exactly?

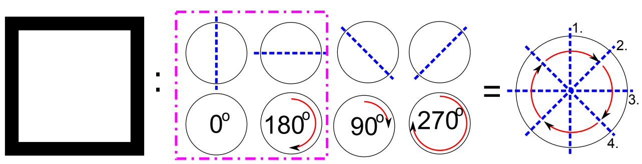


(a) An equilateral triangle (b) An isosceles triangle

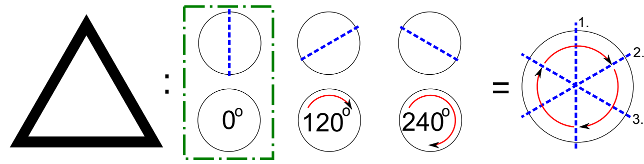
Figure 6: Equilateral triangle and isosceles triangle.

A similar example exists between the equilateral triangle and the isosceles triangle (Fig. 6). An equilateral triangle can be thought of as a special isosceles triangle, and an isosceles triangle can be considered as a distorted equilateral triangle. We would like to say that the equilateral triangle has a “higher” symmetry than the isosceles triangle, but can we make this precise?

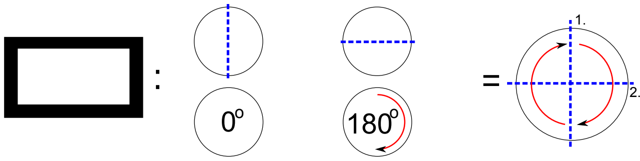
One way to make the preceeding statements precise would be to count the number of symmetry actions in the figures and rank them according to the number of symmetry actions. For the two cases above this indeed seems to work. A square has 8 symmetry actions while a rectangle has 4; An equilateral triangle has 6 symmetry actions while an isosceles triangle has 2 (See Fig 7 for the listing). So in general, the figure with a higher symmetry does have more symmetry actions.



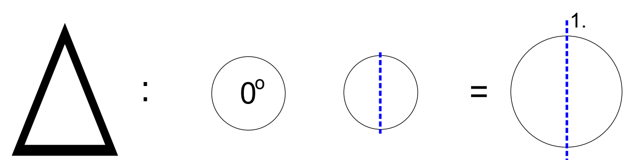
(a) Symmetry of a square



(b) Symmetry of an equilateral triangle



(c) Symmetry of a rectangle

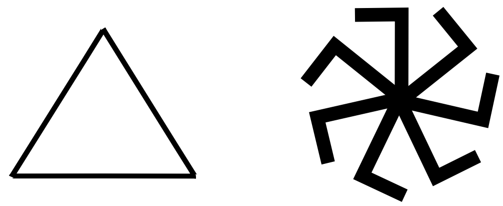


(d) Symmetry of an isosceles triangle

Figure 7: Symmetries of the figures in Fig. 5 and Fig. 6.

Unfortunately, this way of ranking sometimes feels like comparing apples to oranges. For example, consider the equilateral triangle and the seven-leg spiral in Fig. 8. The equilateral triangle has 3 rotations and 3 reflections, while the seven-leg spiral has 7 rotations. If we count the number of symmetry actions, we will conclude that Fig. 8(b) has a “higher” symmetry than the Fig. 8(a). But this doesn’t sound that meaningful at all.

From the list of symmetry actions in Fig. 7. We see another feature among the pairs of figures for which one is considered to have a higher symmetry than the other. Namely, we see that all the symmetry actions of the lower-symmetry figure are all *contained* in



(a) Triangle (b) 7-leg Spiral

Figure 8: Symbols whose symmetry aren't "comparable."

the higher-symmetry one (as indicated by the magenta and green boxes). Thus, we may say that the symmetry exhibited by the lower-symmetry figure is really a *sub-symmetry* of that exhibited by the higher-symmetry figure. More precisely, we define sub-symmetry as follows:

> If all the symmetry actions that preserves figure A also preserves figure B, then we say that the symmetry of figure A is a sub-symmetry of the symmetry of figure B.

Note that the word "sub-symmetry" is always associated with a *pair* of objects or symmetries. We may say something like "the symmetry of a rectangle is a sub-symmetry of that of the square," but never something like "rotation by 90° is a sub-symmetry of the square."

Also, properly speaking I should say something like "the *symmetry* of a rectangle is a sub-symmetry of *the symmetry* of the square," but the sentences will be too long this way, so I may get lazy and say only that "the rectangle is a sub-symmetry of the square."

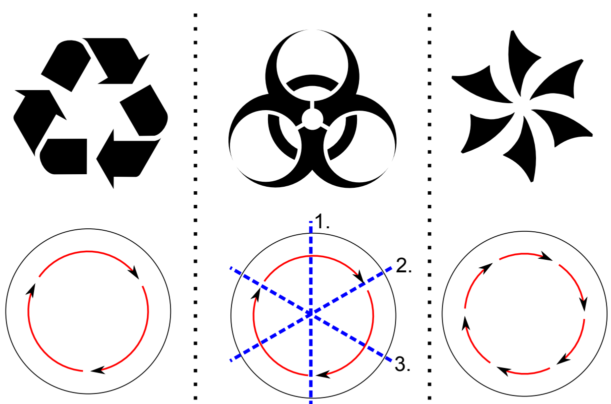
Now that we have a definition, let's practice finding out sub-symmetries in less obvious situations.

Exmaple 1. consider the three symbols in Fig. 9. How does the "sub-symmetry – parent symmetry" relation look like among them?

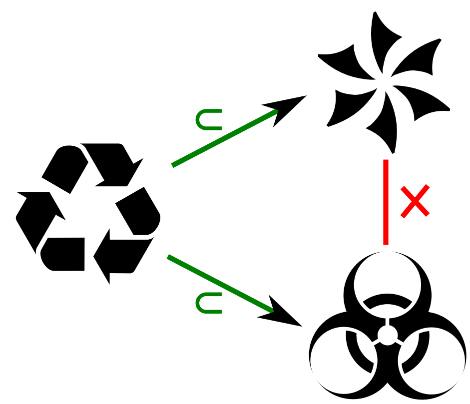
(a) Recycling (b) Biohazard (c) Propeller Asterisk

Figure 9: A set of 3 symbols.

Answer: The symmetry actions of the three figures are listed in Fig. 10(a) using the short-hand notation introduced in session 1. Form this, we see that the symmetry actions of the recycling symbol are rotation by multiples of 120°. These symmetry actions are also present in both the propeller asterisk and the biohazard symbol. Therefore, the symmetry of the recycling symbol is a sub-symmetry of that of the other two.



(a) Symmetries for the symbols in Fig. 9

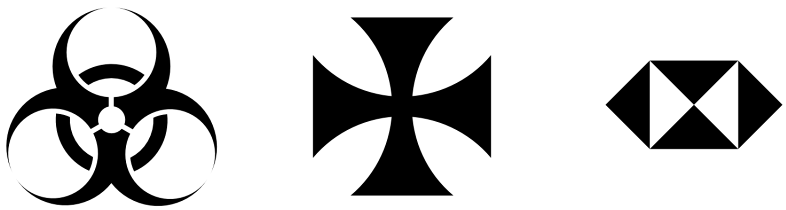


(b) Sub-symmetry relations for the symbols in Fig. 9

Figure 10: Solution to example 1.

On the other hand, the symmetry of the biohazard symbol aren't directly comparable with that of the propeller asterisk, since the propeller asterisk exhibits a 60° rotation symmetry that's absent in the biohazard symbol while the biohazard symbol exhibits a vertical reflection that's absent in the propeller asterisk. The result is summarized in Fig. 10(b), where I have introduced a notation "⊂" used by mathematician to denote sub-symmetry. "$A \subset B$" simply means that A is a sub-symmetry of B. You can think of the symbol $\subset$ as a cousin of the familiar "less than" symbol $<$.

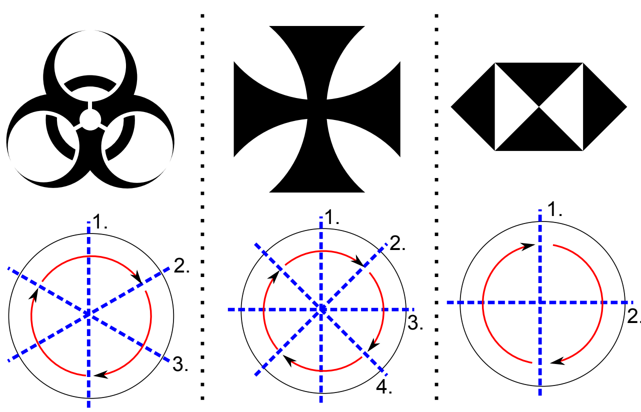
Example 2. consider the three symbols in Fig. 11. How does the "sub-symmetry – parent symmetry" relation look like among them?



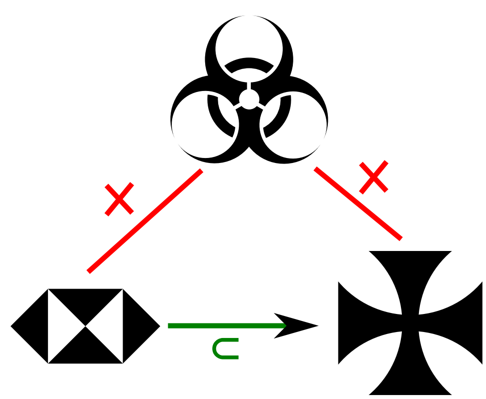
(a) Biohazard (b) Cross Patteée (c) HSBC

Figure 11: Another set of 3 symbols.

Answer: The symmetry actions of the three figures are listed in Fig. 12(a). Observe that the rotations and reflections in the HSBC logo are also present in the cross pattée, thus the symmetry of the HSBC logo is a sub-symmetry of that of the cross pattée.



(a) Symmetries for the symbols in Fig. 11

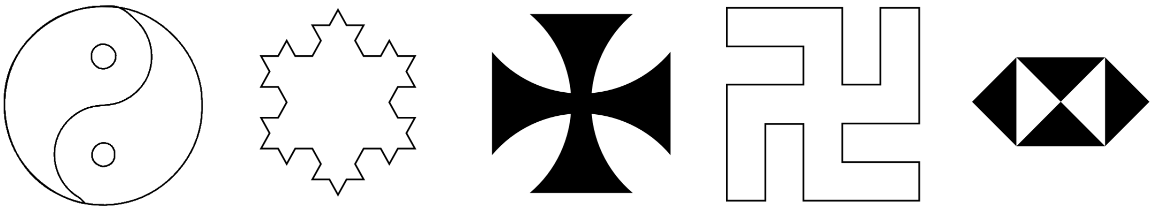


(b) Sub-symmetry relations for the symbols in Fig. 11

Figure 12: Solution to example 2.

On the other hand, the biohazard symbol contains symmetry actions that aren't present in the other two symbols (e.g., rotation by 120° is present in the biohazard symbol but absent in the other two), and vice versa (e.g., the 180° rotation is present in cross pattée and HSBC logo but absent in the biohazard symbol). Therefore, neither the cross pattée nor the HSBC logo are comparable with the biohazard symbol. This result in summarized in Fig. 12(b).

Exercise. Now that we have seen some examples, it's time for you to try. Below are five figures with different symmetries, can you tell which pairs have "sub-symmetry – parent symmetry" relations?



(a) Yin-Yang (b) Kochflake (c) Cross Pattée (d) Swastika (e) HSBC

Figure 13: More symbols that have different symmetries.

We'll go over the answer next time. And if you need some hints, look at the hints on the last two pages....

Image Credits

- Fig. 9(a), Fig. 9(b), Fig. 11, and Fig. 13 are taken or adopted from various pages of Wikipedia. Note that Fig. 11(c) is a company logo.
- Fig. 9(c) is unicode symbol adopted from http://www.fileformat.info/info/unicode/block/mathematical_operators/.

Hint: A Simpler (but Equivalent) Problem

Part of the reason why the problem may be hard is that the figures are fancy and embellished, and so it's not easy to spot their symmetry properties. To help you a bit, here are five alternative figures, which has the same symmetries as the ones used in the problem:

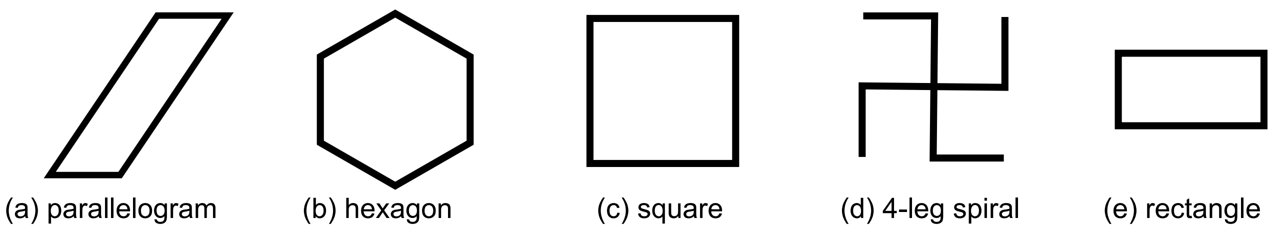
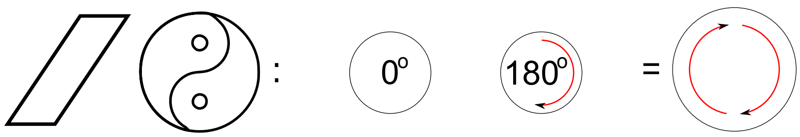


Figure 14: Alternative shapes for the sub-symmetry problem.

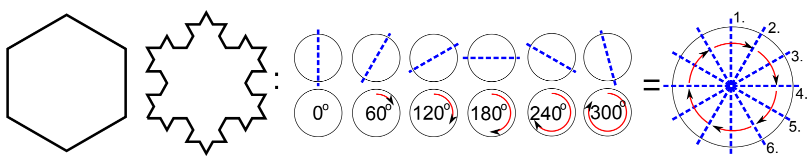
If you still need more helps, flip to the next page....

Hint: Listing the Symmetries

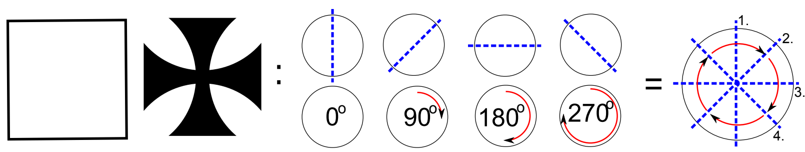
Here is a listing of the symmetry actions for each of the figures in Fig. 13:



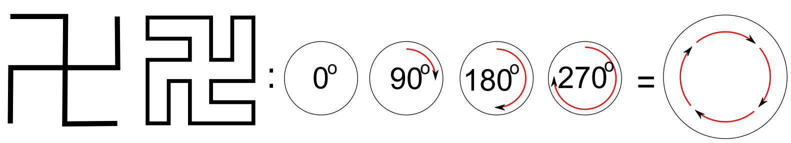
(a) Symmetry of the Yin-Yang



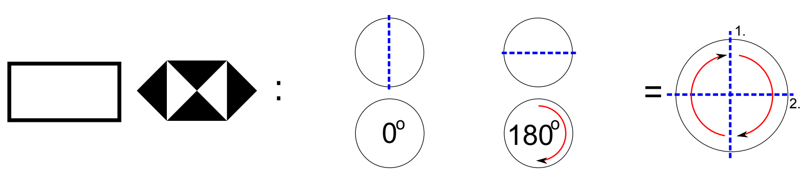
(b) Symmetry of the Kochflake



(c) Symmetry of the Cross Pattée



(d) Symmetry of the Swastika



(e) Symmetry of the HSBC logo

Figure 15: Symmetries of the five figures in Fig. 13.

Now can you tell which pairs have a sub-symmetry – parent symmetry relation?