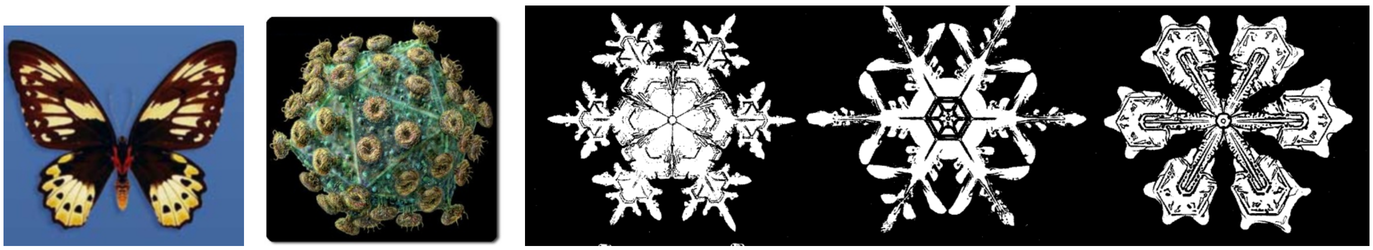
1 Introduction

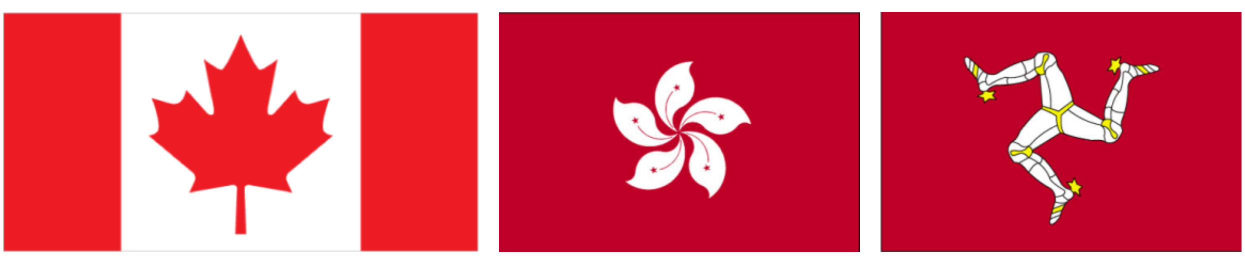
Symmetry is all around us. In Fig. 1(a), we see a butterfly that exhibits a reflection symmetry about the vertical line through the center of its body; In Fig. 1(b), we see a virus that takes the shape of an icosahedron; And in Fig. 1(c), we see snowflakes that possess a six-fold rotation symmetry as well as six distinct reflection symmetries.



(a) Butterfly (b) HIV virus (c) Snowflakes

Figure 1: Symmetries in nature.

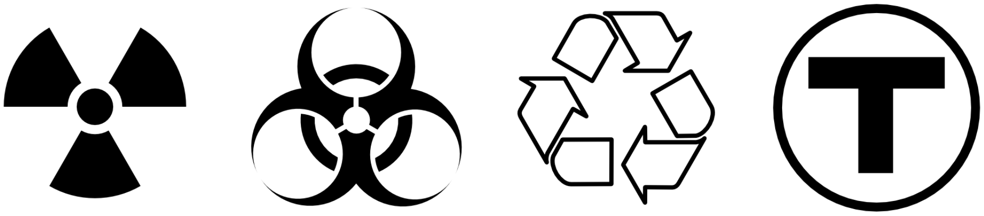
Symmetry is found not only in nature, but also in all kinds of human creations. In Fig. 2 we see symmetry showing up in the flags of various regions, and in Fig. 3 we see symmetry showing up in various signs and logos.



(a) Canada (b) Hong Kong (c) Isle of Man

Figure 2: Flags of various regions.

Why do people want their signs and logos to be symmetric? A quotation from Charles Baldwin, who helped design the biohazard symbol, may be telling:

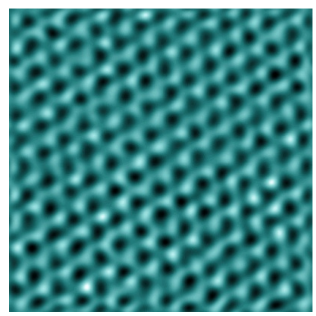


(a) Radiation (b) Biohazard (c) Recycle (d) "Boston T"

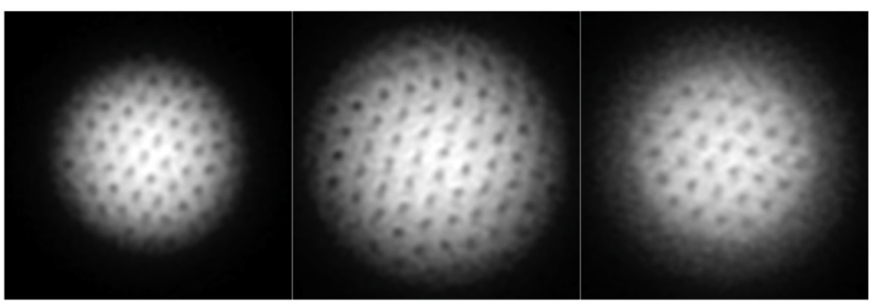
Figure 3: Various signs and logos.

> "We wanted something that was memorable but meaningless, so we could educate people as to what it means." [1]

(a) Honeycomb



(b) Atoms in graphene



(c) A gas of cold atoms

Figure 4: Things that exhibit hexagonal symmetry.

Note that the same symmetry pattern can show up in vastly different phenomena. Fig. 4 is an example. In Fig. 4(a) we see a honeycomb; In Fig. 4(b) we see an image of the atoms in piece of graphene (which is essentially a thin slide of your pencil lead); And in

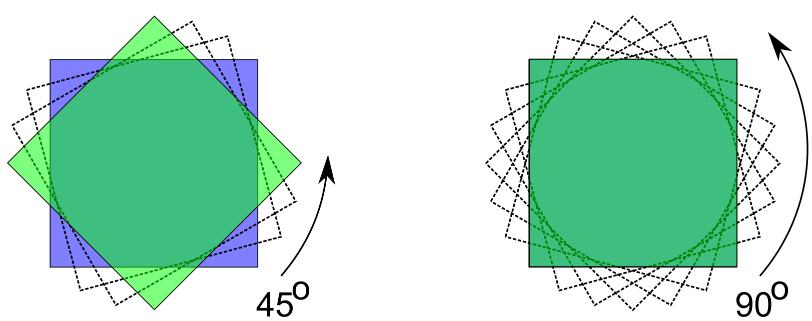
[1] http://www.hms.harvard.edu/orsp/coms/BiosafetyResources/History-of-Biohazard-Symbol.htm

Fig. 4(c) we see a gas of cold atoms being stirred up, with the black dots being vortices ("little tornadoes"). In all three figures we see a regular hexagonal pattern.

Thus, symmetry serves as a unifying framework in understanding nature. That is why it is an interesting subject to mathematicians and scientists alike.

2 Defining Symmetry; Square as an Example

But what exactly do we mean by a "symmetry"? When we say that "rotation by 90° is a symmetry of the square," what exactly do we mean? To understand this, compare a rotation that's a symmetry of the square (e.g., rotation by 90°) with another that isn't (e.g., rotation by 45°). See Fig. 5 for illustration. From the figure, we can see that while rotation by 90° brings the square back to itself (literally "back to square-one"), rotation by 45° doesn't.



(a) Rotate by 45 degrees (b) Rotate by 90 degrees

Figure 5: Action of rotation on a square.

Motivated by this, we can *define* symmetry as follows:

An action is a symmetry of an object if it leaves the object unchanged.

Let's be careful here and note that the word symmetry has multiple but related meanings, depending on context. When we say "rotation by $90°$ is a symmetry of the square," we are referring to a particular action. But we would also say something like "the symmetry of a square consists of four rotations and four reflections," in which case the word symmetry refers to *all* actions that keep the square unchanged. To be careful, I will use the word "symmetric action" to refer to a particular action that keeps an object unchanged, and use the word "symmetry" to refer to the collection of such actions.

Also note that in this precise definition, the word "symmetry" is always associated with an object. For examples, we talk about "the symmetry of a square" or "the symmetry of a triangle." Without an associated object, the word "symmetry" becomes vague. We shall therefore avoid such usage.

The definition of symmetry given above has a few consequences:

1. The "really do nothing" (e.g., rotation by $0°$) action is always a symmetry of an object.
2. One symmetry action followed by another is yet another symmetry action.
3. If a certain action is a symmetry of an object, then the undoing of it is also a symmetry of the object.

For an example of (2), consider again the symmetry of a square. We know that there are precisely eight of them. See Fig. 6 (notice that we identify rotations that differ by $360°$ as the same action. This is because if two rotations are differ by $360°$, then their effect are the same for *all* patterns in space).

According to (2), a rotation by $90°$ clockwise followed by a horizontal reflection should be another symmetry action of the square and thus must be identical to one of the eight actions above. Which one is it? The trick is to test it out on a random (non-symmetric)

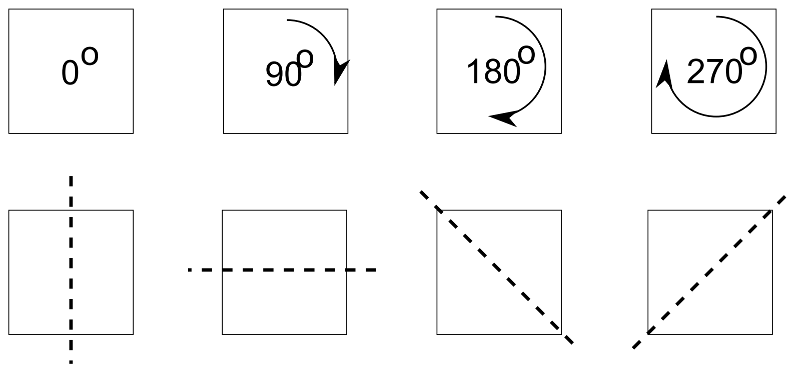


Figure 6: Symmetry of the square.

pattern. By adding a "?" on a square and follow through the two actions, we see that reflecting along the "/" diagonal is the only one among the eight actions that gives the same final pattern. See Fig. 7 for illustration. Therefore, we know that "rotation by 90° clockwise followed by a horizontal reflection" is exactly the same as the "/" diagonal reflection.

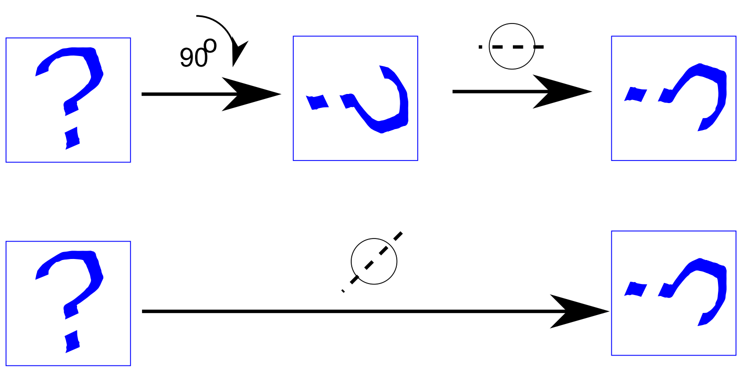


Figure 7: One symmetry action followed by another is yet another symmetry action.

For an example of (3), note that for any random pattern, a rotation of 90° clockwise followed by a rotation of 90° anticlockwise is the same as "really doing nothing." One is thus the "undo" of the other. Therefore, we should always see the two actions in pairs. This is indeed what happened: both of these actions are symmetries of the square, while neither is a symmetry of an equilateral triangle.

Note however that sometimes the "undo" action is the same as the "do" action. For example, to undo a horizontal reflection, you simply do another horizontal reflection.

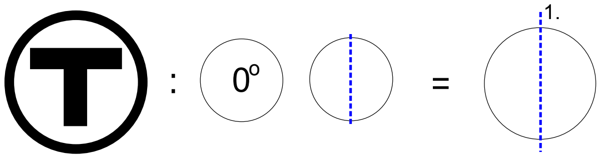
3 Classifying Symmetries of Finite Plane Figures

From Fig. 1–4, we see that while symmetry is a common theme across vastly different phenomena, different objects do exhibit different symmetries (e.g., the symmetry possessed by a square is different from the symmetry possessed by an equilateral triangle). Mathematicians have spent much time classifying all symmetries that one can possibly imagined, and have in fact succeeded to large extent. We couldn't possibly get that far in this class, but we'll nonetheless consider a toy example of such classification. More specifically, we shall classify all possible symmetries of finite plane figures.

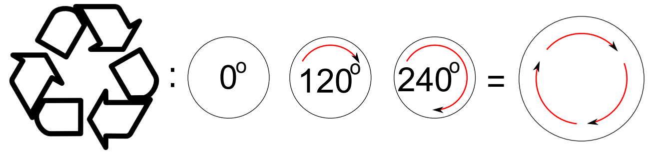
Let's go back and look at Fig 3 more carefully. What symmetry does each figure have? The answer:

- The radiation sign Fig. 3(a) and the biohazard sign Fig. 3(b) have the same symmetry, which consists of a 3-fold rotation plus three distinct reflections.
- The symmetry of the recycling sign Fig. 3(c) consists of a 3-fold rotation.
- The symmetry of the "Boston T" sign Fig. 3(d) consists of one reflection plus the "really do nothing" 0-degree rotation.

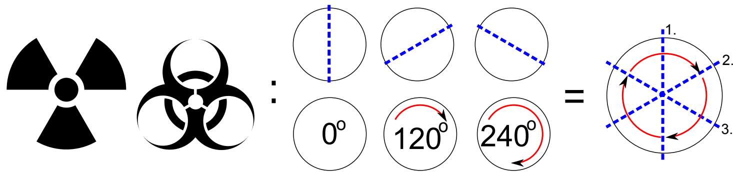
For illustrations, see Fig. 8. I have also introduced a short-hand notation (on the right of the figures) to specify the symmetry actions of an object all at once.



(a) Symmetry of the "Boston T" sign



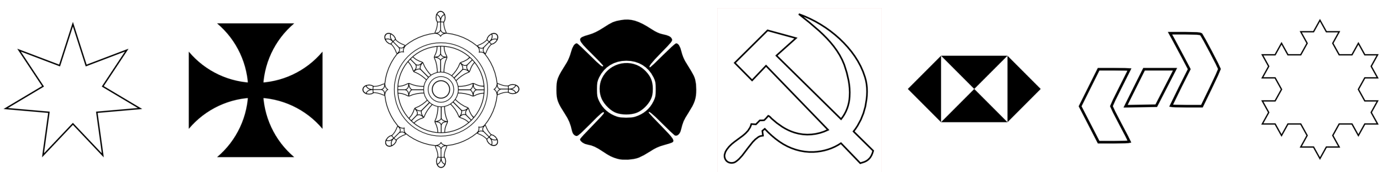
(b) Symmetry of the recycling sign



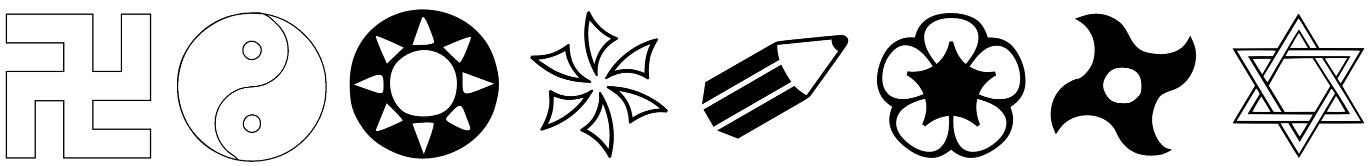
(c) Symmetry of the radiation sign and the biohazard sign

Figure 8: Symmetry of the signs and logos in Fig. 3.

Now, let's repeat the exercise for a host of figures:



(a) (b) (c) (d) (e) (f) (g) (h)



(i) (j) (k) (l) (m) (n) (o) (p)

Figure 9: A bunch of logos.

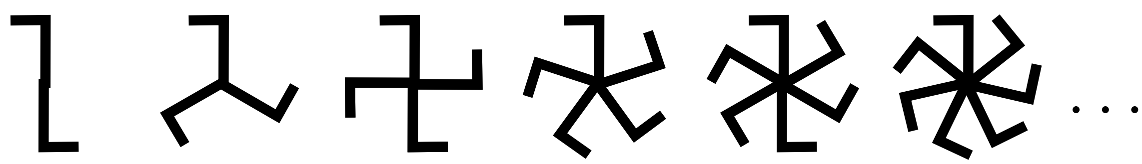
(The answer is on the very last page. Please try before looking at the answer.)

If we look at the symmetries exhibit in these figures more carefully, we can see that they can be grouped into two main classes. The first class are symmetries that consist only of rotations; the second class are symmetries that consist of both rotations and reflections. Moreover, if we count the "really do nothing" action as a rotation (which is a rotation by

0 degree), then the second class of symmetries always have same number of reflections and rotations. These observations are in fact generally true. In other words:

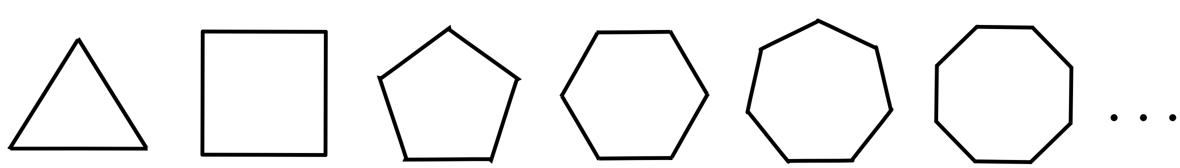
> The symmetries of all finite plane figures[a] fall into one of the two categories:
>
> **(a)—The Cyclic.** The symmetry consists of entirely of n-fold rotation (i.e., rotation by $360/n$ degrees, and its multiples).
>
> **(b)—The Dihedral.** The symmetry consists of an n-fold rotation together with n distinct reflections.
>
> [a]except those that have infinitely many symmetry actions, e.g., the symmetry of a circle.

We can see the general patterns in each of the categories above by considering their prime examples. The cyclic symmetries can be visualized in "windmill" patterns, while the dihedral symmetries can be visualized in regular polygons. See Fig. 10 and Fig. 11 for reference.



(a) C_2 (b) C_3 (c) C_4 (d) C_5 (e) C_6 (f) C_7

Figure 10: The prime examples of cyclic symmetries.



(a) D_3 (b) D_4 (c) D_5 (d) D_6 (e) D_7 (f) D_8

Figure 11: The prime examples of dihedral symmetries.

The statement that the cyclic and the dihedral are the only symmetries for finite plane figures can be *proved* rigorously. The full proof will be too dry and a bit hard for us.

But we can get a feeling of how it works by considering two sample questions. These, incidentally, are applications of the properties of symmetry that we learned previously.

Q1. Why can't there be a symmetry of finite plane figures that have (two or more) reflections but no (non-trivial) rotations?

NB. The terms under the brackets are there to rule out special cases. Since we count "really do nothing" as a rotation, we put "non-trivial" to neglect it; and since it is possible to have a figure whose symmetry consists only of one reflection in addition to the "really do nothing," we put "two or more" to rule out this special case.

A1. Suppose a finite plane figure has two distinct reflections. According to property (2) of symmetry, by doing one reflection followed by the other, we obtain yet another symmetry action of that figure. It can be checked that one reflection followed by a different one always result in a non-trivial rotation. Thus it is impossible to have a finite plane figure whose symmetry consists of two or more reflections but no non-trivial rotations.

Q2. For the dihedral, why is the number of reflections always the same as the number of rotations?

A2. Let's suppose we start with one specific reflection. For each of the non-trivial rotation, doing that rotation follow by the specific reflection will produce another reflection. It turns out that these new reflections will each be along a different axis. It also happens that all reflections can be generated this way. Hence the number of reflections must equal to the number of rotations.

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- Fig. 1(a): Taken from http://www.scienceinschool.org/2006/issue2/symmetry
- Fig. 1(b): Taken from http://www.healthinitiative.org/HTML/hiv/firstcontact/hivbig.htm
- Fig. 1(c): Taken from W. A. Bentley's *Snow Crystals*.
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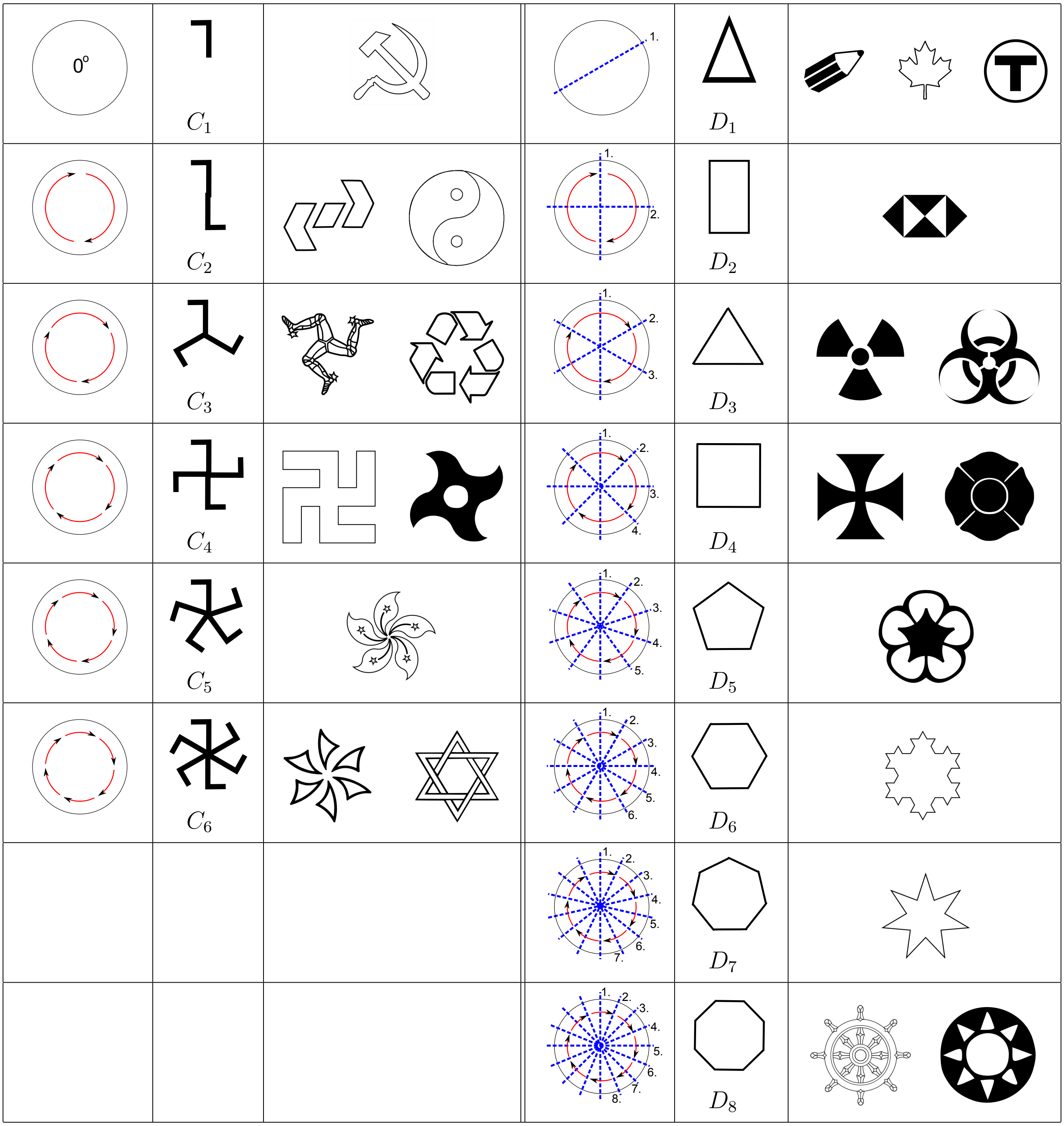
- Fig. 3: Taken from various pages of Wikipedia.
- Fig. 4(a): Taken from Wikipedia.
- Fig. 4(b): Taken from http://www.its.caltech.edu/~yehgroup/stm/images.html
- Fig. 4(c): Taken from the imgage gallery of Ketterle's (MIT) research group (http://cua.mit.edu/ketterle_group/experimental_setup/BEC_I/image_gallery.html)
- Fig. 9: (a)–(j) are taken or modified from Wikipedia. Note that (f) and (g) are company logos. (k)–(n) are unicode symbols and are taken from http://www.fileformat.info/info/unicode/block/dingbats/images.htm. (o) is adopted from the google cache of www.astraware.com and (p) is adopted from http://www.thecolor.com/Category/Coloring/Hanukkah.aspx

Answer to the classification problem.



In the above I have used the shorthand notation introduced in Fig. 8. Note that for completeness I also included the shapes from Fig. 2 and Fig. 3).

Note that I grouped the pencil logo, the Canada maple logo and the “Boston T” logo together. For the symmetry of each consists of a single reflection plus the “really do nothing” action, despite the fact that the reflection axis of the pencil is at a different angle then the Canada maple and the “Boston T”.