Solving the Cubic Equation

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1.1 Problem-Solving Strategies

1. **Rational root theorem**: Let a, b, c, d be integers. If $ax^3 + bx^2 + cx + d = 0$ has a rational root r, then the numerator of r divides d and the denominator divides a.

2. **Synthetic division**: The example shows $x^3 + 6x^2 + 4x - 8$ divided by $x + 2$ is $x^2 + 4x - 4$.

-2	|	1	6	4	-8
	|		-2	-8	8
	$-$	$-$	$-$	$-$	$-$
		1	4	-4	0

3. **Reducing a cubic**: For any cubic $ax^3 + bx^2 + cx + d = 0$ with integer coefficients, the substitution $x = \frac{y-b}{3a}$ allows you to obtain a cubic equation in y with no y^2 term, leading term y^3, and integer coefficients.

4. **Discriminant:** For the cubic $x^3 - mx - n = 0$, define $\Delta = \left(\frac{m}{3}\right)^3 - \left(\frac{n}{2}\right)^2$. The reduced cubic has three distinct real roots if $\Delta > 0$, three real roots with one double root if $\Delta = 0$, and one real root and two complex conjugate roots if $\Delta < 0$.

5. **Reducible case**: Suppose $\Delta \leq 0$. Choose real numbers p, q such that $p + q = n$ and $pq = \left(\frac{m}{3}\right)^3$. Then $x = \sqrt[3]{p} + \sqrt[3]{q}$ is a solution to the cubic. The other two solutions are $\omega \sqrt[3]{p} + \omega^2 \sqrt[3]{q}$ and $\omega^2 \sqrt[3]{p} + \omega \sqrt[3]{q}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$ is a cubic root of unity. Don't forget to convert back from the reduced form to the original form!

6. **Irreducible case**: Suppose $\Delta > 0$. Find θ such that $\cos 3\theta = \frac{n/2}{(m/3)^{3/2}}$. Then the solutions are $2\sqrt{\frac{m}{3}} \cos(\theta + \frac{2k\pi}{3})$, where $k = 0, 1, 2$.

Example. To solve the cubic $x^3 - 12x + 16 = 0$, we compute $\Delta = 64 - 64 = 0$, so we proceed as in Reducible case. We find p, q such that $p + q = -16$ and $pq = 4^3 = 64$. By inspection, we can choose $p = -8$ and $q = -8$. Thus $\sqrt[3]{p} = -2$ and $\sqrt[3]{q} = -2$, so the solutions are

$$(-2) + (-2) = -4, (-2)\omega + (-2)\omega^2 = 2, (-2)\omega^2 + (-2)\omega = 2.$$

1.2 Computations

Find all solutions to the given equations. Use Wolfram-Alpha to check your answers. (Question 9 was from the 2014 AIME.)

1. $3x^3 + 81 = 0$
2. $x^3 - 18x - 30 = 0$
3. $x^3 + 3x^2 + 15x + 1 = 0$
4. $x^3 + 8x + 24 = 0$
5. $x^3 - 12x - 8 = 0$
6. $2x^3 - 6x^2 + 36x - 27 = 0$
7. $3x^3 + 11x^2 + 2x - 2 = 0$
8. $27x^3 + 27x^2 - 1 = 0$
9. $-8x^3 + 3x^2 + 3x + 1 = 0$
10. $(x^3 + 3x^2 - 2)^2 - 3 = 0$

1.3 Problems

Please justify your answers to the best of your ability.

1. Suppose a cubic polynomial $f(x) = x^3 - mx - n$ has a double root r.
   (a) Show that the roots of f are r, r, and $-2r$.
   (b) Find a formula for r in terms of m, n.

2. (a) Let $\omega = \frac{-1+i\sqrt{3}}{2}$. By expanding the right-hand side, show that

   $$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega).$$

   (b) Find the roots of $x^3 - 6x + 9$.
   (c) How does this identity relate to finding the roots of a cubic?

3. Verify Problem-Solving Strategy 4.

4. (a) Compute the value of $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$.
   (b) Show that $\sqrt[3]{3 + \sqrt{10}} + \sqrt[3]{3 - \sqrt{10}}$ is irrational.

5. We will try to find a cubic polynomial with roots $r_1 = \cos\frac{2\pi}{7}, r_2 = \cos\frac{4\pi}{7}, r_3 = \cos\frac{6\pi}{7}$.
   (a) Let $x = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ be a primitive seventh root of unity, i.e. $x^7 = 1$ and $x \neq 1$. Show that $1 + x + x^2 + \cdots + x^6 = 0$.
   (b) Use $\cos x = \cos(2\pi - x)$, the previous result, and DeMoivre's theorem to show that $r_1 + r_2 + r_3 = -\frac{1}{2}$.
   (c) Use product-to-sum relations to compute $r_1r_2 + r_2r_3 + r_3r_1$ and $r_1r_2r_3$.
   (d) Find a cubic polynomial with integer coefficients and roots r_1, r_2, r_3.
   (e) Try computing its roots using the standard methods. What do you find?