From Zero To Infinity... And Beyond!

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The **objective** of this course is to explore concepts from set theory, analysis, and combinatorics, leading up to a discussion about the different degrees of infinity.

1 Week 1: Constructing the Natural Numbers

1.1 Introduction to the class and to HSSP

1.2 Sets: Some basic definitions and notation

When you start from nothing, from scratch, the first intuitive thing to try to quantify is having something: this concept is called a **set**. Discuss definitions of **collection**, **subset**, the **axiom of extension**, **contained in**, **not contained in**, and **empty set**.

1.3 Equalities of sets and new set construction

Make some simple sets using words and explain the nuances involved in this. Construct some sets using specifications via functions. Discuss again the equality of sets and the un-intuitiveness of some examples.

1.4 Russell's paradox

Show how a specific set construction implies that there is no universe.

1.5 Actually construct the natural numbers

Define **successor sets** and discuss the **axiom of infinity**. Now that all the mechanics are in place, actually construct the naturals. This construction will serve as a definition (explain the consequences of this).

1.6 Extra stuff

If there is time, talk about how arithmetic on the natural numbers is defined. This is straightforward once all the definitions above have been covered. Perhaps move this into the beginning of the next class. Or it can be omitted altogether.

2 Week 2: Constructing the Real Numbers

This week will be a slight digression from the axiomatic nature of the rest of the class. It will focus more on an analysis-type digression into the construction of the real numbers. The method discussed will be the method of Dedekind cuts. If the members of the class want to see more set theory instead, we can vote and choose to skip/modify this section to have a more set theoretic feel. My goal, however, is to explore the numbers themselves rather than a specific branch of math related to the numbers.

2.1 Definitions

Define **upper bound**, **least upper bound**, **ordering**, and **ordered field**.

2.2 The Actual Dedekind Cuts

Actually go through and define the different components of Dedekind cuts that will be used to construct the real numbers. The actually construct some real numbers using this method.

2.3 Extra stuff

If there is time, construct the real numbers from the method of Cauchy sequences used in analysis. This requires some more intensive work with definitions, but depending on what people in the class want, this may be more interesting to them.

3 Week 3: Infinities based on Cardinality

3.1 Definitions and notation

Define **cardinality**, **injection**, **surjection**, and **bijection**. Discuss some notation. [10 mins]

3.2 Define the cardinality of $\mathbb{N}$

Define the cardinality of $\mathbb{N}$, make some examples of bijections, and talk about the meaning of cardinality in terms of a type of equivalence.

3.3 Cardinality of $\mathbb{R}$

Prove that no injection exists between $\mathbb{N}$ and $\mathbb{R}$. Define the cardinality of $\mathbb{R}$ as something different than the cardinality of $\mathbb{N}$.

3.4 Power sets

Talk about power sets and their relation to cardinality. Prove that the power set of a set is always of a higher cardinality.

3.5 Define cardinal numbers

Actually define what cardinal numbers are based on the previous discussion. [15 mins]

3.6 Extra stuff

Start discussing cardinal arithmetic OR discuss the continuum hypothesis (or generalized continuum hypothesis), explain why it is a hypothesis, and explain why it is consistent with the axioms of set theory but explain why it is not a theorem. To decide this, take a vote to see what the students want.

4 Week 4: Infinities based on Ordinality

4.1 Definitions

Define what ω is and where the concept of *ordinal* comes from. Then define successors of ω.

4.2 Some ordinal numbers

List some ordinals and talk about the Burali-Forti paradox (there cannot be a set consisting of all ordinal numbers).

4.3 Induction and transfinite induction

Talk about induction vs. transfinite induction and when such arguments could be useful/necessary. Prove how and why it works. Invoke things we learned about cardinal and ordinal numbers.

4.4 Ordinal arithmetic

Define ordinal sums, the intuitiveness of adding infinities, properties of ordinal addition, ordinal products, associativity, properties of ordinal products, associativity, distributivity, and commutativity.

4.5 Examples

Lots of examples! Lots of examples! Time permitting, more examples!

5 Week 5: Cardinal and Ordinal Arithmetic

5.1 Cardinal arithmetic

Addition, multiplication, exponentiation, the intutitiveness of these operations

5.2 Nuances

Cases where arithmetic is done with natural numbers and infinities; the intuitiveness of this.

5.3 The relation between cardinal and ordinal numbers

Cantor's paradox (there is no largest cardinal), notational issues, paradoxes with notational ambiguities, weird arithmetic you can do with these issues.

5.4 Even more examples!

This whole section is peppered with examples, but do even MORE examples!

5.5 Extra stuff

Peano axioms, ZF set theory, or analysis, based on what the students want. I will have each of these things prepared so I can teach any of them. Decide this by vote.