Slinky whistlers

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Delightful sounds, produced in simple experiments, agree with an elementary theoretical analysis.

INTRODUCTORY EXPERIMENTS

Get a slinky.[1] Use tape or staples to fasten one end, the output end, flat against a wall (or door or window), to serve as a sounding board. Hold the other (input) end in your hand and stand about ten feet from the wall, far enough so that the slinky is suspended with neighboring turns not in contact with one another, but close enough to one another so that there is still negligible tension in the coiled spring. Now tap the input end with a pencil and listen. Radiating from the output at the wall you will hear a "whistler," a sound that first becomes audible as a very high pitch and then quickly descends in pitch, becoming inaudible in a fraction of a second. Evidently, the high frequency components of the "delta function" excited by the pencil tap propagate down the slinky delay line more rapidly than do the low frequency components.[2]

Try standing farther from the wall, increasing your distance to it by about a factor of 2. This will put some tension in the slinky. However, the whistler sound (frequency versus delay time) coming from the wall does not change. Now increase the distance again, so that the spring is acquiring some tension. The sound is still unchanged. (With one slinky I walked nearly 100 ft from the wall, completely ruining the slinky as I stretched and unwound it irreversibly; nevertheless, the whistlers recorded on my tape recorder at the wall did not change appreciably during the process.) Thus the spring tension has very little to do with the whistler we hear.

Next, shorten the slinky delay line by a factor of 2 by gathering in half of the turns at the input end and holding them in your hand. The whistler becomes more rapid, its delay before audible onset and its duration both decreasing by a factor of 2. Continue shortening the delay line by a factor of 2 each time. The whistler becomes shorter and shorter until, with only a few turns left, it becomes a nondispersed "click."

Now return to the full length, but instead of holding the input end in your hand (which acts as an absorbing termination for reflected waves) hang the input on a nail on the wall opposite the sounding board. For a single input tap you will now hear several repeated whistlers, each weaker than the previous one, and with a larger dispersion (larger time duration), as the wave packet reflects back and forth from one end to the other. (This is analgous to the ionospheric whistlers of Refs 2 and 3.)

THERE ARE TWO WHISTLER MODES

Next (holding the input end in your hand again, so as to damp reflections) notice that there are two different ways you can excite the slinky so as to produce two distinctly different-sounding whistlers. If you tap the slinky in the radial direction, so that the pencil tends to bend the wide transverse dimension of the slinky wire, you hear a "fast" whistler. If instead you tap along the helix axis (by inserting the pencil between two adjacent turns before you tap) so as to bend the thin dimension, you hear a "slow" whistler. To my ear the ratio of time durations of these two whistler modes is about a factor of 2. To check this hypothesis, I first excite the fast whistler (by tapping the wide dimension in the radial direction), using the full N-turn delay line. Then I gather N/2 of the turns into my hand (so that the remainder gives a delay line half as long) and excite the slow whistler (with a longitudinal tap along the thin dimension). My ear cannot distinguish between these two whistlers. To the extent that they are indeed identical, there really *is* a factor of 2 in time duration (for the same slinky length). Since the ratio d_1/d_2 is 4.1, which we can call 4, we conclude that any successful slinky–whistler theory must predict a sound velocity in the slinky that goes as the square root of the thickness of the transverse dimension d that gets bent by the wave.

THEORY

We need a theory that can be compared with experiment. The simplest possibility is that the whistler we hear is not influenced by the fact that the wire is wound in a spiral. (That hypothesis is supported by the "ruined slinky" experiment mentioned above.) We therefore assume the whistler is simply due to transverse traveling waves in a stiff long straight rectangular steel bar under no tension, having cross-sectional dimensions d_1 and d_2 and length L, with the numerical values given in Ref. 1. The dispersion relation between frequency $f = \omega/2\pi$ and wavelength $\lambda = 2\pi/k$ for transverse waves in such a bar is given (See Appendix I or Ref. 5.) by

$$\omega = k^2 cr, \tag{1}$$

where $r = d/(12)^{1/2}$, d is the bar thickness in that transverse dimension which is being deformed by the wave, r is the radius of gyration for rotation of a thin transverse slice of wire about an axis lying in the plane of the slice perpendicular to dimension d and passing through the center of the slice, and $c = (Y/\rho)^{1,2}$ is the velocity of the nondispersive *longitudinal* (compressional) sound waves in the bar, which is made of material having Young's modulus Y and density ρ. Then the wave velocity $v = \omega/k$ and group velocity $v_g = d\omega/dk$ are given by

$$v = \omega/k = kcr = [\omega cr]^{1/2}, \quad v_g = 2v. \tag{2}$$

PREDICTIONS OF THE THEORY

Notice that Eq. (2) satisfies the experimental result found above, that the sound velocity is proportional to $d^{1/2}$. Good! Also, the higher frequencies travel faster than the lower, as observed. Good!

The time delay t between the delta function excitation at your hand at time zero and the arrival of a wave packet of dominant frequency f at the sounding board, a distance L along the wire from your hand, is given by $t = L/v_g$ (*not* L/v). Taking handbook values of Y and ρ for steel gives $c = 5.08 \times 10^5$ cm/s. For my slinky, with $L = 2180$ cm, Eq. (2) then gives

$$t = L/v_g = 1.14/f^{1/2}, \tag{3}$$

with t in s, f in H, and d in cm. For the fast whistler we take $d = d_1 = 0.262$ cm. For the slow whistler $d = d_2 = (1/4)d_1$, as given above.[1]

COMPUTER SIMULATION EXPERIMENT

In order to compare Eq. (3) with what I hear, I programmed my Macintosh to emit a sound of frequency f as a function of delay time t as given by Eq. (3), with $d = d_1$ or d_2, and with t starting, not at zero, but at an initial time t_0 such that Eq. (3) gives an initial frequency $f_0 = 20$ kHz, close to the limit of human hearing; each successive value of f is emitted for a duration of 1/60 s. To my ear the sound of the Mac whistler matches very well the sound of either the slow or the fast slinky whistler from my wall sounding board. This is especially true if I cut off the Mac sound after about 0.25 s, thus cutting off the low frequency tail of the whistler. Part of the reason for the absence of low frequencies emitted by the wall was the poor low frequency coupling of the slinky to the wall, as shown by the fact that by holding my end of the slinky very close to my ear I could hear a much longer whistler (longer than the expected twice-as-long duration due to the replacement of L by the down-and-back distance $2L$), which descended to much lower frequencies than those heard coming from the wall. They sounded like Mac whistlers with a 1 s cutoff.

MORE REAL EXPERIMENTS

Encouraged by the good match between the Mac whistlers and the real slinky whistlers, I set up an oscilloscope at home, with two microphones, the first attached to the slinky at its input end, where it could pick up the delta function so as to trigger the scope, the second at the output end, to pick up and display the dispersed delta function—the whistler. I stapled the output end of the slinky to the wooden wall of my dining room, and supported the input end with a hook screwed into the opposite wall. I also replaced my hand by a large glob of "duct seal" wrapped around the first few turns at the input end, so as to absorb most of the wave reflected back from the wall.

The first observations with the oscilloscope confirmed what my ear had heard, namely that there are two modes differing by a factor of 2 in velocity. This was easily seen on the oscilloscope because of the unexpected peculiarity that each of the modes appeared to have both a high and a low frequency cutoff, which gives corresponding early and late delay time cutoffs, according to Eq. (3). Thus the fast whistler, excited by tapping in the radial (wide) direction, was mostly contained between time delay $t = 0.015$ and 0.035 s, with maximum amplitude near 0.030 s, whereas the slow whistler, excited by tapping in the longitudinal (thin) direction, had maximum amplitude near $t = 0.060$ s and was mostly contained between $t = 0.050$ and 0.090 s. Thus, surprisingly, there was very little overlap in delay times of the two whistler modes. [No such cutoffs or separations in time are predicted by Eq. (3).] With practice in tapping it was possible to excite the fast whistler mode with very little of the slow, as indicated on the scope by the time durations just mentioned, and by my ear, and to excite the slow mode without too much of the fast.

In order to test Eq. (3) I operated the oscilloscope such that the signal from microphone no. 2 (at the wall) was displayed in a magnified "time window," of total length 1 ms (100 μs/division) which could be delayed by a variable amount t after the scope trigger at time zero provided by microphone no. 1 at the input end. I used a Polaroid Land camera to record the oscilloscope trace from the microphone no. 2 in the time window, so as to measure the average periods of a few cycles and hence measure the oscillation frequency f of the dominant Fourier component at delay t. With the delay set at $t = 0.015$ s, Eq. (3) predicts $f = 22$ kHz for the fast whistler, and 88 kHz for the slow. When I excited the fast whistler I found a fairly pure sine wave in the window, with $f = 22$ kHz within 10% accuracy. The theory works! When I excited the slow whistler I also found 22 kHz in the window, but with much smaller amplitude. Thus no matter how I excited the slinky I found only 22 kHz, the expected fast whistler frequency at this delay. Thus we have good agreement with the theory, except that the theory expects both modes to be present at all times. Why do I not see the expected 88 kHz signal at $t = 0.015$ s, even when I tap longitudinally? It could be because of poor coupling to the wall, or poor response of the microphone. Of course, its absence in the delayed magnified window also corresponds to what I see with normal triggering (for which I can see the whistler amplitudes but cannot read their frequencies because I am then using a slow sweep with 10 ms per scope division)—namely, very little of the slow whistler appears before $t = 50$ ms delay. Incidentally, the fine agreement between theory and experiment for the fast whistler mode proves that it is indeed the group velocity, not the wave velocity, that goes into Eq. (3). (Did you doubt it? If so, you are in good company.)

With the window delay set at $t = 0.030$ ms the predicted values are $f = 55$ kHz for the fast whistler and 22 kHz for the slow. For either excitation method I found 5.0 kHz, with a much larger amplitude when I tapped radially to excite the fast mode. Frequency $f = 5.0$ kHz corresponds to the fast whistler ($d = 0.262$) with delay $t = 31.3$ ms, according to Eq. (3). That is in good agreement with my measured setting of $t = 0.030$ s, since I determine this window delay by reading the location of an intensified beam spot on the slow scale of 10 ms per scope division, and I can only read this to $\pm$ ms. So again I get beautiful agreement with Eq. (3) for the fast whistler. But again I see no hint of the expected 22 kHz slow whistler signal, even when I tap longitudinally.

With the 1 ms window delay set at $t = 55 \pm 1$ ms, Eq. (3) predicts $f = 1.6$ kHz for the fast whistler mode and 6.5 kHz for the slow. When I excited the slow mode I measured 6.3 kHz in the window, which corresponds to $t = 0.056$ s for the slow whistler, according to Eq. (3). So now I have beautiful agreement with the *slow* whistler! They *both* agree with Eq. (3)! But I could see no signal at 1.6 kHz when I tried to excite the fast whistler. Thus I get good agreement with the slow whistler and no hint of the fast, at $t = 55$ ms. Of course, these were simple "eyeball" frequency measurements, made by counting three or four of the dominant nearly sinusoidal oscillations, on a Land photograph of the scope trace in the delayed window. Small admixtures of other Fourier components could not be found in this way.

SPECULATIONS AND LOOSE ENDS

Why do we get such good agreement between Eq. (3) and both whistler modes? Is it reasonable for these transverse waves not to "notice" that they are "going in circles" on a helical slinky, rather than traveling along a straight bar as the theory assumes? One turn of the slinky helix has length 23 cm. If a given frequency f has a wavelength $\lambda = v/f$ (not v_g/f) that is much smaller than 23 cm it would seem reasonable for the wave not to notice. Now, the longest times where I saw significant amounts of the fast whistler on the scope were at $t_{\max} \sim 40$ ms, corresponding, accordng to Eq. (3), to $f_{\min} \sim 3.1$ kHz, and therefore to $\lambda_{\max} = v/f_{\min} \sim 9$ cm. The corresponding quantities for the slow whistler were $t_{\max} \sim 90$ ms, $f_{\min} \sim 2.4$ kHz, and $\lambda_{\max} \sim 5$ cm. Both of these values of $\lambda_{\max}$ are sufficiently small compared with 23 cm that it seems reasonable that the waves not notice the helical path. That, I believe, is why we can get such good agreement with Eq. (3), for the observed wavelengths, which are all less than these $\lambda_{\max}$. Is it then the "noticing they are going in circles" that is the cause of the observed long wavelength cutoffs? I do not think so. Instead, I believe the cutoffs are simply due to poor coupling of the slinky to the microphone, as observed above when I put the slinky close to my ear. It would be worthwhile to improve this coupling so as to see if the beautiful agreement between Eq. (3) and experiment continues to hold at lower frequenceis, where λ approaches or exceeds 23 cm. (But the analysis would become less simple if the present fortuitous "decoupling in time" between the two modes were lost. With both modes present at all times, one would have to do "honest" Fourier analysis, instead of simply counting sine waves on a Land photograph as I did. We leave this for the student.)

What is the source of the observed high frequency cutoffs? For the fast whistler I see on the scope $t_{\min} \sim 0.015$ s giving $f_{\max} \sim 22$ kHz and $\lambda_{\min} \sim 3.3$ cm; for the slow whistler I see $t_{\min} \sim 0.050$ s giving $f_{\max} \sim 8$ kHz and $\lambda_{\min} \sim 2.8$ cm. The fact that these two $\lambda_{\min}$ are nearly equal, whereas the corresponding $f_{\max}$ are not, suggests that this cutoff is really due to a length, not a time. One possibility may be due to the fact that the input deformation caused by the pencil tap is not a delta function but perhaps has an effective extent of about 3 cm. That is plausible because of the fact that the helix radius of 3.6 cm is suspiciously close to these values of $\lambda_{\min}$. Other suspects are the coupling of the slinky to the microphone via the wall, and the response of the microphone. Use of a less primitive coupling of the microphone and slinky (perhaps by cementing the slinky to the moveable element of an audio loudspeaker) and exploration of the low and high frequency cutoffs would be intersting student projects. Are the cutoffs real, or are they the result of the crudeness of my technique? Do the whistlers propagate for wavelengths greatly in excess of 23 cm? For what wavelengths does Eq. (3) break down? We leave these loose ends as exercises for the student.

de BROGLIE WHISTLERS

Finally, it is intersting to note that the slinky dispersion relation Eq. (1), $\omega = Ak^2$, with A a constant, has the same form as the dispersion relation for de Broglie waves associated with a nonrelativistic free particle, where we have $E = p^2/2m$, $E = \hbar\omega$, $p = \hbar k$, and therefore $\omega = Ak^2$, with $A = \hbar/2m$. Just as for slinky whistlers, the group velocity (particle velocity) of a de Broglie wave packet is twice the phase velocity. However, the phase velocity ω/k of a de Broglie wave is a rather nebulous quantity: It is unobservable, since ω can be changed to $\omega + \omega_0 = \omega + V_0/\hbar$, with no physical consequences, by simply adding a constant potential energy V_0 (at all locations) to the energy E. By contrast, for slinky whistlers ω is a very physical quantity. You can *hear* it.

APPENDIX: DERIVATION OF THE DISPERSION RELATION EQ. (1)

(For an alternative derivation see Ref. 5). Consider a stiff wire of rectangular cross section, lying along the z axis at equilibrium, and having transverse dimension $2l$ along x. We think of the wire as made of repeated infinitesimal modules of length a along z, and width $2l$ along x, as shown in Fig. 1, where we show modules number $n-1$, n, and $n+1$.

Module number n extends from $z = -a/2$ to $+a/2$ and $x = -l$ to $+l$. Each module is built on a rigid massless "cross" with a horizontal arm of length a along z, and a vertical leg of height $2l$ along x. The vertical leg of each cross has a mass m at each end, at distances $x = \pm l$ from the Z axis. (Later we will add more masses, distributed uniformly from $x = -l$ to $+l$.) The masses on neighboring modules are connected by horizontal springs of length a and spring constant K, that are neither compressed nor extended, at equilibrium. The ends of the horizontal arms of neighboring modules are joined by frictionless slotted hinges, (shown as hollow circles in Fig. 1). The slots (not shown) allow the joined ends to slide freely with respect to one another in the z directon, but the horizontal springs oppose such sliding. The hinges allow rotation of the modules in the z-x plane. The constraint force that prevents separation along x, at the hinges, is a shearing force S. When all the modules are at their equilibrium positions, lined up along the z axis as shown in the Fig. 1, the shear force S is zero. When S is not zero it is a slowly varying function of z. When that is the case the modules have slight bends at the hinges, gradually changing the module angle with increasing z. The "wavelength" for such changes is large compared to the module length a. In Fig. 1 we show the shearing force S as the force in the $+x$ direction on the right-hand end of the horizontal arm of module n, and S' as the force in the $+x$ direction on the right-hand end of the horizontal arm of module $n-1$. (The figure does not show the slight progressive bending at the hinges that should accompany the presence of a shear.) Because of Newton's third law, the force on the left-hand end of the horizontal arm of module n is $-S'$. Thus the net force on module n, in the x direction, is $S - S'$. (The springs contribute forces only in the z direction.)

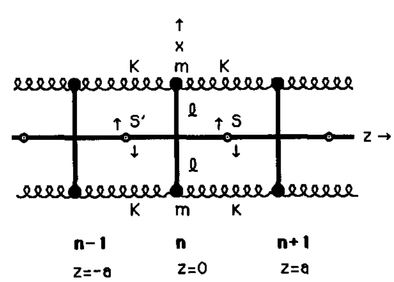

Fig. 1. Lumped-parameter model of a stiff spring. The length "a" of one module is small compared with any of the wavelengths involved.

Let us first consider the simplest motion, where the relative motion of the modules is entirely along z. Then there is no shear force S. The force is entirely due to compression or extension of the horizontal springs. Let $\psi_z(n,t)$ be the z displacement of module n from its equilibrium position, and $\psi_z(n+1,t)$, that of module $n+1$. Then the net z force on module n due to the two right-hand springs is $2K[\psi_z(n+1)-\psi_z(n)]$. In the continuous limit this becomes $2Ka\partial\psi_z/\partial z$, evaluated at $z=a/2$. Similarly the two left-hand springs give a net z force of $-2Ka\partial\psi_z/\partial z$ at $z=-a/2$. The total force due to all four springs is thus, in the continuous limit,

$$F_z = 2Ka^2\partial^2\psi_z/\partial z^2, \quad \text{(A1)}$$

evaluated at $z=0$. According to Newton's second law this gives, for the z acceleration of the module of mass $M=2m$,

$$(2m)\partial^2\psi_z/\partial t^2 = F_z = 2Ka^2\partial^2\psi_z/\partial z^2, \quad \text{(A2)}$$

which can be written

$$\partial^2\psi_z/\partial t^2 = c^2\partial^2\psi_z/\partial z^2, \quad \text{(A3)}$$

where

$$c^2 = Ka^2/m. \quad \text{(A4)}$$

Here c^2 is the square of the wave velocity for the longitudinal waves satisfying the classical wave equation, Eq. (A3). For a definite angular frequency ω these waves have the form (for traveling waves)

$$\psi_z = \mathrm{A}\cos(kz-\omega t+\alpha), \quad \text{(A5)}$$

with dispersion relation given by substituting (A5) into (A3) to get

$$\omega^2 = c^2k^2. \quad \text{(A6)}$$

For these waves the wave velocity ω/k and group velocity $d\omega/dk$ both equal c for all k; therefore all frequencies travel at the same speed c. Thus a delta function excitation of longitudinal waves does not become "dispersed" so as to become a whistler.

For a wire of rectangular cross section A we can imagine it to be made up of a large number N of the above one-dimensional wirelets, bundled together. Since the springs are in parallel, the total spring constant is larger than K by a factor N, but so is the mass, so the ratio K/m in Eq. (A4) is unchanged. Multiply numerator and denominator of Eq. (A4) by area A. define $m/Aa=\rho$, the mass per unit volume, and $Y=Ka/A$, Young's modulus, so that Eq. (4) gives $c^2=Y/\rho$, which we can look up in the handbook, given the material.

Next consider the case of present interest, where the motion is transverse. Then the shear force will be non zero. Let $S_x(z)$ be the shear force in the $+x$ direction on the right-hand end of a horizontal arm that has its right-hand end located at z. Then the central module of Fig. 1 has total force F_x in the x direction given by $F_x = S_x(a/2) - S_x(-a/2)$. When we go to the continuous limit and assume $S_s(z)$ is a slowly varying continuous function of z that gives

$$F_x = \frac{a\partial S_x}{\partial z}. \quad \text{(A7)}$$

Let $\psi_x(n,t) = \psi_x(z,t)$ be the x displacement of module n from equilibrium at time t. Then Newton's second law [with Eq. (A7)] gives

$$\frac{M\partial^2\psi_x}{\partial t^2} = F_x = \frac{a\partial S_x}{\partial z}, \quad \text{(A8)}$$

where $M=2m$ for the sketch. Notice that in the configuration shown in the sketch both the shear forces S_x have the same lever arm $a/2$ and exert essentially the same torque (same direction and nearly the same magnitude) on the central module, so that the net counterclockwise torque on the central module due to shear is aS_x. We do not want our module to start rapidly rotating. We therefore assume that whenever there is a shear S_x, as shown, the modules will have reached a quasistatic equilibrium where there is a counterbalancing torque due to the horizontal springs such that the total torque is zero. (It is not precisely zero. We allow gradual changes in module angle as long as the changes are negligible in the small change in z, $\Delta z = a$.) That implies a gradually changing rotation angle of the modules. Let θ_n be the clockwise angle that the vertical leg of module n makes with the vertical x axis. (We assume $\theta_n \ll 1$ rad.) The elongation of the top-right spring is then $l(\theta_{n+1}-\theta_n)$. Multiply that elongation by K to get the spring tension, and then by l to get the clockwise torque $Kl^2(\theta_{n+1}-\theta_n)$ exerted about the center of module n by the top-right spring. The bottom right spring is compressed by the same amount that the top-right spring is extended, and doubles this torque to $2Kl^2(\theta_{n+1}-\theta_n)$. In the continuous limit we have $\theta_n+1-\theta_n = a\partial\theta/\partial z$, evaluated at $z=a/2$, so the torque due to the two right-hand springs becomes $2Kl^2a\partial\theta/\partial z$, evaluated at $z=+a/2$. The two left-hand springs similarly contribute a clockwise torque $-2Kl^2(\theta_n-\theta_{n-1}) = -2Kl^2a\partial\theta/\partial z$, at $z=-a/2$. All four springs therefore give a total clockwise torque $2Kl^2a^2\partial^2\theta/\partial z^2$, at $z=0$. We demand that this torque cancel the counterclockwise torque aS_x due to shear, so we must have

$$aS_x = \frac{2Kl^2a^2\partial^2\theta}{\partial z^2}. \quad \text{(A9)}$$

Then Eq. (A8), with $M=2m$, combined with (A9) gives

$$m\frac{\partial^2\psi_x}{\partial t^2} = \frac{Kl^2a^2\partial^3\theta}{\partial z^3}. \quad \text{(A10)}$$

But, we also have, for slowly varing θ,

$$\theta = -\frac{\partial\psi_x}{\partial z}, \quad \text{(A11)}$$

so that (A10) becomes

$$\frac{\partial^2\psi_x}{\partial t^2} = -\frac{(Kl^2a^2/m)\partial^4\psi_x}{\partial z^4}. \quad \text{(A12)}$$

Now we add additional masses m distributed uniformly from $x=-l$ to $+l$ between the two masses shown in Fig. 1. We connect the new masses to their horizontal neighbors by additional horizontal springs of constant K. Since the springs are in parallel, their spring constants are additive, but so are the masses m; therefore the ratio K/m in Eq. (A12) is unchanged. However l^2 in Eq. (A12) must be replaced by the average of x^2 from $x=0$ to $x=l$. That replaces l^2 by $(1/3)l^2$, which is called r^2, where r is the "radius of gyration." Thus Eq. (A12) becomes [using also Eq. (A4)]

$$\frac{\partial^2\psi_x}{\partial t^2} = -\frac{c^2r^2\partial^4\psi_z}{\partial z^4}, \quad \text{(A13)}$$

where $r^2 = l^2/3$, and c^2 is given by Eq. (A4). Equation (A13) is a wave equation, but it is not the "classical" one. For a definite angular frequency ω a traveling wave solution to Eq. (A13) has the form

$$\psi_z = A\cos(kz - \omega t + \alpha). \tag{A14}$$

Substituting (A14) into the (A13) gives the dispersion relation

$$\omega^2 = c^2 r^2 k^4, \tag{A15}$$

which is Eq. (1).

[1]My slinky was made by James Industries, Inc., Hollidaysburg, PA, and is available in many toy stores. It is made of steel wire having a rectangular cross section. Mine has wire with wide transverse dimension $d_1 = 0.262$ cm and thin transverse dimension $d_2 = 0.064$ cm. The wire is wound in a helical coil of diameter 7.3 cm, so that each turn has length $\pi \times 7.3 = 23$ cm along the wire. The thin transverse dimension is oriented along the helix axis, and the wide dimension along the radial direction, with successive wide flat faces resting against one another when the slinky is not extended. My slinky has $N = 95$ turns for a total wire length $L = 95 \times 23 = 2180$ cm, and a total helix length $95 \times 0.064 = 6$ cm when unextended. Some toy stores sell plastic slinkies that give fine whistlers. There are also smaller-diameter steel slinkies.

[2]The most famous whistlers are electromagnetic; a lightning stroke "delta function" in the Earth's northern hemisphere produces radiation that travels out through the ionosphere along the Earth's magnetic lines of force as if in a wave guide, returns to the Earth's surface in the southern hemisphere, reflects there and retraces its path, returning once more to the surface near its starting point, where it can be heard on an ordinary audio amplifier equipped with a 200-ft antenna. Because of the dispersive behavior of the ionosphere the high-frequency components of the delta function return first, and one hears a swiftly descending whistle. See Ref. 3. (For an entirely different kind of whistler see Ref. 4)

[3]R. Helliwell, *Whistlers and Related Ionospheric Phenomena* (Stanford U. P., Stanford, CA, 1965).

[4]F. S. Crawford, "Culvert whistlers," Am. J. Phys. **39**, 610 (1971).

[5]R. V. Sharman, *Vibrations and Waves* (Butterworth, London, 1963), Sec. 5.7. Sharmon's Eq. (5.34) is equivalent to our Eq. (1), but Sharmon's "k" is our $r = d/(12)^{1/2}$, the radius of gyration.

The Casimir effect revisited

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Using Wick's normal-ordered expression for the quantum-mechanical energy of a radiation field, so that in absence of matter the zeropoint energy of a radiation field would vanish, the Casimir effect of attraction between uncharged condenser plates in absence of photons will still follow, due to a nonvanishing but finite zeropoint radiation energy in the presence of these plates. If we assume the plates to be transparent for electromagnetic waves of sufficiently high frequency, the remaining zeropoint energy density itself will be finite (and not merely its variation under a change of the distance between the plates). These results are obtained by a more careful consideration of where in sums over modes of vibration one should use sums over running waves with a periodicity condition, where one should sum over standing waves with nodes at conducting boundaries, and where these sums may be replaced by integrals.

I. INTRODUCTION

In its most primitive form, quantum field theory of elementary particles expresses physical quantities for bosons by classical field expressions and for fermions by the expressions of wave mechanics, replacing in these expressions the fields by so-called "quantized" fields, i.e., by operators which describe the annihilation and creation of elementary particles, when they operate on the state vector Ψ, which for these fields of particles describes the probabilistic state, somewhat like in elementary wave mechanics the wave function ψ describes the probability distribution of an electron.[1] These classical expressions in terms of quantized fields generally lead to nonsensical (infinite) results. Presently we will not consider the more sophisticated infinities that arise from interactions between particles, or the methods by which those infinities may be eliminated. More trivial infinities arise already when one considers the total electric charge of, say, a free electron field, or the total energy of a field of noninteracting bosons. These infinities are trivial, because it is easy to simply omit them, in Dirac's electron wave mechanics by ignoring the charge of infinitely many electrons occupying states of negative energy, and in quantum electrodynamics by writing for the energy of free photons

$$E^0 = \sum_k N_k^0 h\nu_k^0, \tag{1a}$$

instead of

$$E_c^0 = \sum_k (N_k^0 + \tfrac{1}{2}) h\nu_k^0, \tag{1b}$$

where the subscript c reminds us of the use of the classical expression for the energy expressed in terms of quantized fields, while the superscripts 0 remind us that we here are considering a boson field in the absence of matter. With a finite number of free bosons present (or with none present when $N_k^0 = 0$), we could formally calculate E^0 from the quantized classical E_c^0 by subtracting from the latter the

