A **vector space** V over a field F has the operations of **addition** and **scalar multiplication**, and satisfies several basic laws. A vector space in a vector space is a **subspace**.

A vector $v \in V$ is a **linear combination** of vectors of $S \subseteq V$ if there exist a finite number of vectors $u_1, u_2, \ldots u_n \in S$ and scalars $a_1, a_2, \ldots a_n \in F$ such that

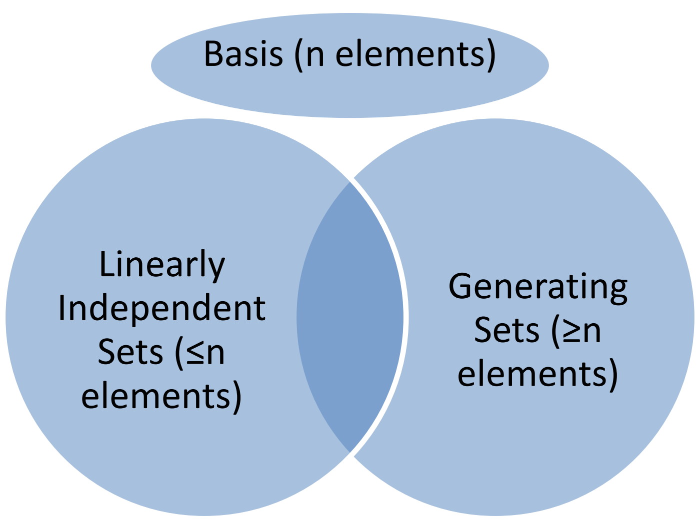
$$v = a_1 u_1 + \cdots + a_n u_n.$$

If 0 can be nontrivially written in this form, S is **linearly dependent**. The set of all v in the above for is the subspace **generated** (spanned) by S.

A **basis** β for V is a linearly independent subset of V that generates V.

<u>Replacement Theorem</u>: (Simplified) Every linearly independent set can be made into a basis by adding elements.

Every basis for V contains the same number of vectors. The unique number of vectors in each basis is the **dimension** of V (dim(V)).



Every vector space has a basis.

For vector spaces V and W over F, a function $T: V \to W$ is a **linear transformation** (homomorphism) if for all $x, y \in V$ and $c \in F$,

(a) $T(x + y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

The **null space** or kernel is the set of all vectors x in V such that T(x)=0.

$$N(T) = \{x \in V | T(x) = 0\}$$

The **range** or image is the subset of W consisting of all images of vectors in V.

$$R(T) = \{T(x) | x \in V\}$$

Both are subspaces. **nullity**(T) and **rank**(T) denote the dimensions of N(T) and R(T), respectively.

<u>Dimension Theorem</u>: If V is finite-dimensional, nullity(T)+rank(T)=dim(V)

Linear transformations (over finite-dimensional vector spaces) can be viewed as left-multiplication by matrices, so linear transformations under composition and their corresponding matrices under multiplication follow the same laws. This is a motivating factor for the definition of matrix multiplication. Facts about matrices can be proved by using linear transformations, or vice versa.

Matrix product:

Let A be a $m \times n$ and B be a $n \times p$ matrix. The product AB is the $m \times p$ matrix with entries

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}\ , 1 \leq i \leq m, 1 \leq j \leq p$$

Interpretation of the product AB:

1. Row picture: Each row of A multiplies the whole matrix B.
2. Column picture: A is multiplied by each column of B. Each column of AB is a linear combination of the columns of A, with the coefficients of the linear combination being the entries in the column of B.
3. Row-column picture: C_{ij} is the dot product of row I of A and column j of B.

The matrix representation of T in $\beta = \{v_1, \ldots, v_n\}$ and γ is $A = [T]_\beta^\gamma$. Load the coordinates of $T(v_i)$ into the ith column. $[I_V]_\beta^\gamma$ changes β-coordinates to γ-coordinates. So:

$$[T]_\gamma = [I_V]_\gamma^\beta [T]_\beta [I_V]_\beta^\gamma$$
$$B = QAQ^{-1}$$

Linear transformations T, U	Matrices A, B
rank(TU) ≤ min(rank(T), rank(U))	rank(AB) ≤ min(rank(A), rank(B))

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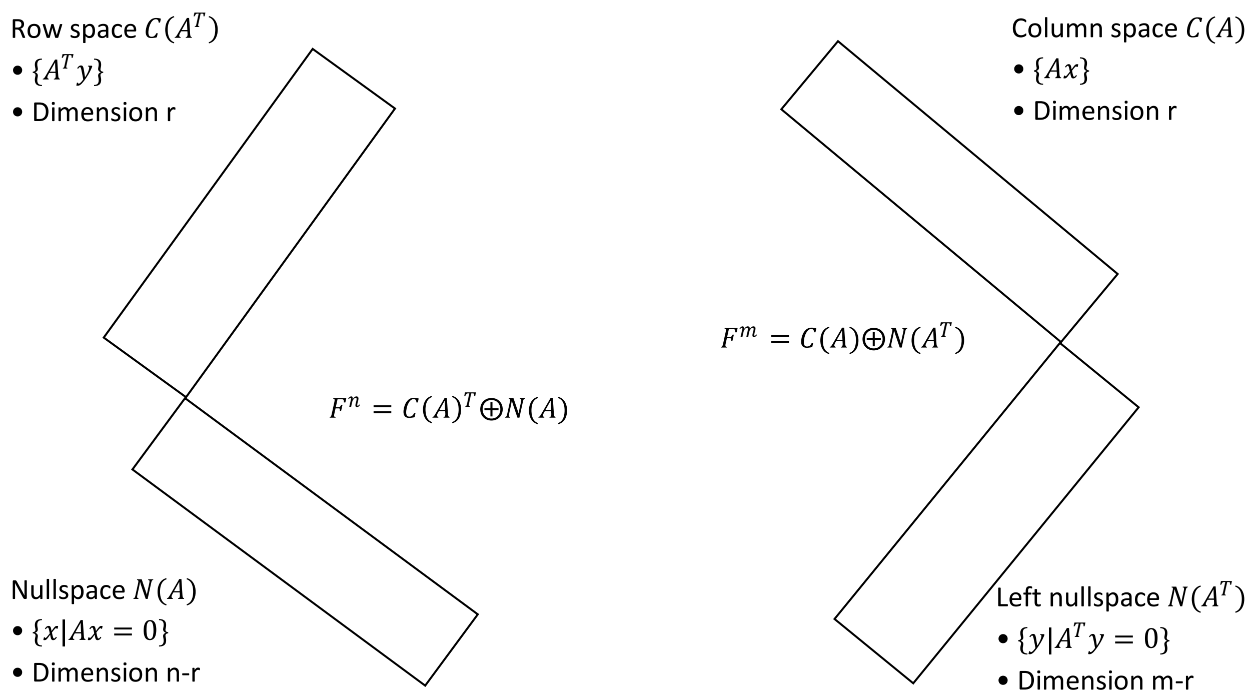
<u>Fundamental Theorem of Linear Algebra (Part 1)</u>:

Dimensions of the Four Subspaces: A is mxn, rank(A)=r (If the field is complex, replace A^T by A^*.)



The **determinant** (denoted $|A|$ or $\det(A)$) is a function from the set of square matrices to the field F, satisfying the following conditions:

1. The determinant of the nxn identity matrix is 1, i.e. $\det(I) = 1$.
2. If two rows of A are equal, then $\det(A) = 0$, i.e. the determinant is alternating.
3. The determinant is a linear function of each row separately, i.e. it is n-linear. That is, if $a_1, \dots a_n, u, v$ are rows with n elements,

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

These properties completely characterize the determinant.

Cofactor Expansion: Recursive, useful with many zeros, perhaps with induction.
(Row)

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij})$$

(Column)

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is A with the *i*th row and *j*the column removed.

<u>Cramer's Rule</u>:

If A is a nxn matrix and $\det(A) \neq 0$ then $Ax = b$ has the unique solution given by

$$x_i = \frac{\det(B_1)}{\det(A)}, 1 \leq i \leq n$$

Where B_i is A with the *i*th column replaced by b. If $\det(A) = 0$, then A is singular (has no inverse).

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Let T be a linear operator (or matrix) on V. A nonzero vector $v \in V$ is an **eigenvector** of T if there exists a scalar λ, called the **eigenvalue**, such that $T(v) = \lambda v$. The **eigenspace** of λ is the set of all eigenvectors corresponding to λ: $E_\lambda = \{x \in V | T(x) = \lambda x\}$.

The **characteristic polynomial** of a matrix A is $\det(A - \lambda I)$. The zeros of the polynomial are the eigenvalues of A. For each eigenvalue solve $Av = \lambda v$ to find linearly independent eigenvalues that span the eigenspace.

If there are n linearly independent eigenvalues, T (A) is diagonalizable:

$$[T]_\gamma = [I_V]_\gamma^\beta [T]_\beta [I_V]_\beta^\gamma$$
$$A = Q \Lambda Q^{-1}$$

Where $\Lambda = [T]_\beta$ is a diagonal matrix.

Applications to recursive sequences, probability (Markov chains).

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The **incidence matrix** of a graph: A has a row and column for each vertex, and $A_{ij} = 1$ if vertices i and j are connected by an edge, and 0 otherwise.

The incidence matrix A for a family of subsets $\{S_1, \dots, S_n\}$ containing elements $\{x_1, \dots, x_m\}$ has $A_{ij} = \begin{cases} 1 \text{ if } x_i \in S_j \\ 0 \text{ if } x_i \notin S_j \end{cases}$. Exploring AA^T and using properties of ranks, determinants, linear dependency, etc. may give conclusions about the sets. Working in the field $\mathbb{Z}_2$ on problems dealing with parity may help.