

Numbers That Do Weird Things, Week 3

HSSP Summer 2016

2016-07-24

(U) Denotes a problem that will build on your basic understanding and (hopefully) help you gain intuition for the topic. Expect these problems to be easier.

1. 2D Euclidean Distance

Show that the Euclidean distance in 2 dimensions,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is a metric.

2. Strong Triangle Inequality

Prove that $|x + y|_p \leq \max(|x|_p, |y|_p)$ for $x, y \in \mathbb{Q}$. This inequality is called the *strong triangle inequality*.

3. Convergent Sequences are Cauchy

A sequence (a_0, a_1, \dots) *converges* to a limit L if for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. In other words, the elements of the sequence are getting arbitrarily close to L . Prove that every convergent sequence is a Cauchy sequence.

4. (U) Practice with the P-Adic Norm

For each of $p=2,3,5$, find the p -adic norm of the following numbers: 2, 3, 5, $\frac{32}{25}$, $\frac{17}{11}$, 36, $\frac{15}{128}$, 120. Make a picture depicting the relative sizes of the numbers for each of the p -adic norms.

5. (U) Understanding the P-Adic Norm

What does it mean for some rational q to satisfy $|q|_p \leq 1$ (Explain in words)?

6. Practice with Bases

(a) $23_b = 32_c$ and $15_b = 22_c$. Find the base numbers b and c

(b) Convert $\frac{1}{5}$ to binary.

(c) Consider the sequence 1, 5, 6, 25, 26, 31, ... consisting of sums of distinct powers of 5. What is the 50th term of the sequence?