Week 1: Goodstein Sequences and Ordinal Numbers

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1.1 Goodstein Sequences

To write a number in *base* b, we write it as a sum of powers of b, with fewer than b copies of the same power. For instance, we write 2,760,403 in base 10 as

$$1{,}760{,}403 = 10^6 + 10^5 \cdot 7 + 10^4 \cdot 6 + 10^2 \cdot 4 + 3.$$

The last term is $10^0 \cdot 3$, which we write as 3 since $10^0 = 1$. We write 333 in base 2 as

$$333 = 2^8 + 2^6 + 2^3 + 2^2 + 1.$$

To write a number in *hereditary base* b, we write it in base b and then write all of the exponents in hereditary base b. In our first example, the exponents are all less than 10, so there's nothing left to do. To write 367 in hereditary base 2, we need to repeat the process three times:

$$333 = 2^8 + 2^6 + 2^3 + 2^2 + 1 = 2^{2^3} + 2^{2^2+2} + 2^{2+1} + 2^2 + 1 = 2^{2^{2+1}} + 2^{2^2+2} + 2^{2+1} + 2^2 + 1.$$

At the end, every aspect of the number is written as a sum of powers of b. There should be no 'plain' numbers larger than b left over.

The *Goodstein sequence* for some number n is computed as follows:

1. The first term is n. Write n in hereditary base 2.
2. Replace every 2 with a 3, and subtract 1. This is the second term; write it in hereditary base 3.
3. Replace every 3 with a 4, and subtract 1. This is the third term.
4. Continue in this way: to compute the next term, increase the base by 1 by replacing each b with $b+1$, and subtract 1. Then write the number in hereditary base $b+1$.

For instance, the Goodstein sequence for 333 begins

	b	Hereditary base b	Value
1	2	$2^{2^{2+1}} + 2^{2^2+2} + 2^{2+1} + 2^2 + 1$	367
2	3	$3^{3^{3+1}} + 3^{3^3+3} + 3^{3+1} + 3^3$	443426488243037769948249836510281987560
3	4	$4^{4^{4+1}} + 4^{4^4+4} + 4^{4+1} + 4^3 \cdot 3 + 4^2 \cdot 3 + 4 \cdot 3 + 3$	$\approx 3.2 \cdot 10^{66}$

It gets to some very large numbers! Not that in the third row, it would have ended $4^4 - 1$, but this isn't valid hereditary basy 4 because it uses subtraction, so we replace it with $4^3 \cdot 3 + 4^2 \cdot 3 + 4 \cdot 3 + 3$. This is analogous to $10{,}000 - 1 = 9{,}999$.

Let's do a full example with smaller numbers: the Goodstein sequence for 3. In hereditary base 2, we have $3 = 2 + 1$, so the Goodstein sequence is

	b	Hereditary base b	Value
1	2	$2+1$	3
2	3	3	3
3	4	3	3
4	5	2	2
5	6	1	1
6	7	0	0

We say that a Goodstein sequence *terminates* when it reaches zero. This one terminates after just six terms. Once the base is larger than the current value, replacing b with $b+1$ does nothing, so we just subtract 1 each term and eventually get down to zero.

Since the Goodstein sequence for 3 was kind of boring, let's compute the Goodstein sequence for $4 = 2^2$, which starts

	b	Hereditary base b	Value
1	2	2^2	4
2	3	$3^2 \cdot 2 + 3 \cdot 2 + 2$	26
3	4	$4^2 \cdot 2 + 4 \cdot 2 + 1$	41
4	5	$5^2 \cdot 2 + 5 \cdot 2$	60
5	6	$6^2 \cdot 2 + 6 + 5$	83

Exercise 1.1. Do you think the Goodstein sequence for 4 will terminate? If so, how long will it take? Make a guess, and then compute the next several terms. You can skip terms if you see a pattern to save time, and there's no need to check the exact value of every term (just leave them in hereditary base b). (The answer will be revealed later.)

There's an obvious question about Goodstein sequences:

Question 1.2. Does every Goodstein sequence terminate?

We could compute Goodstein sequences, but this quickly becomes tedious, and won't tell us about all of them. We'd like a mathematical argument that either shows some specific Goodstein sequence goes forever, or that all of them terminate. This turns out to require the ordinal numbers.

1.2 Ordinal numbers

The ordinals numbers are what you get if you count forever, and then keep on counting. We start counting with the natural numbers:

$$0, 1, 2, 3, \ldots.$$

After counting all infinitely many natural numbers, we want to keep counting. We'll call the next number, which is infinitely large, ω ('omega'). Then we keep counting

$$\omega, \omega+1, \omega+2, \ldots.$$

You might object that ω is infinity, and you've heard somewhere that infinity plus one is infinity. That's true in some contexts, but in the ordinal numbers ω and $\omega+1$ and different infinite ordinals. If we said $\omega = \omega + 1$, then there wouldn't be anywhere to keep counting after ω.

After everything of the form $\omega + n$, the next ordinal is $\omega + \omega = \omega 2$. So we continue

$$\omega 2, \omega 2 + 1, \ldots, \omega 3, \ldots, \omega 4.$$

I'm being careful to write $\omega+1$ and $\omega 2$ instead of $1+\omega$ and 2ω, because ordinal arithmetic isn't commutative: $1 + \omega = \omega \neq \omega + 1$ and $2\omega = \omega \neq \omega 2$. This won't matter much for today, but we can talk about it in a future week.

After all ordinals of the form ωn, we have $\omega \cdot \omega = \omega^2$. We just keep going in the same way, as long as we possibly can:

$$\omega^2, \ldots, \omega^2+\omega, \ldots, \omega^2+\omega 2, \ldots, \omega^2+\omega^2 = \omega^2 2 \ldots, \omega^2 3, \ldots, \omega^3, \ldots, \omega^4, \ldots, \omega^\omega, \ldots, \omega^{\omega+1}, \ldots$$

$$\omega^{\omega 2}, \ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^3}, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^2}}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots, \omega^{\omega^{\omega^{\omega^\omega}}}, \ldots.$$

After all the ordinals of this form, we reach an infinite power tower of ω's, which we call ε_0:

$$\varepsilon_0 = \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\cdots}}}}}}$$

We could keep going, eventually reaching ordinals like $\varepsilon_0^{\varepsilon_0}$, and $\varepsilon_0^{\varepsilon_0^{\varepsilon_0}}$, and the next infinite tower $\varepsilon_1 = \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}}$. And then we could keep going some more, which we might do later. But for today, the ordinals up to ε_0 are enough.

1.2.1 The ordinals are well-ordered

The most important property of the ordinal numbers is that they are *well-ordered*. This means either of these equivalent statements:

1. Any (nonempty) collection of ordinal numbers has a least element.
2. Any strictly decreasing sequence of ordinal numbers must eventually reach 0.

Conversely, the real numbers are *not* well-ordered:

Exercise 1.3. Describe a collection of real numbers which has no least element, and a strictly decreasing sequence of real numbers which goes on forever. Can you use only nonnegative real numbers?

We haven't defined ordinals clearly enough to formally prove that they're well-ordered, but we can intuitively explain why they must satisfy each formulation of well-orderedness:

1. If you count up from 0, counting infinitely through the ordinals, there must be some first time where the number you count is in the collection. That number is its least element.
2. Suppose for instance we start with ω^2, and name a decreasing sequence of ordinals. The next ordinal has to be less than ω^2, so it must be $\omega \cdot a + b$ for some finite a and b. If we keep counting down, eventually b must reach 0, at which point we have to 'break up' the $\omega \cdot a$, replacing it with $\omega \cdot a'$ for some smaller a'. Continuing, eventually we have to reach 0. You can make a similar argument for starting from a larger ordinal than ω^2: from ω^3, the first step must give a finite multiple of ω^2, which we know must reach 0, and so on. You can think of this like playing a game (in fact, this game can be embedded in Chess, which we might see later) where each turn you name a smaller ordinal, and you want to survive as long as possible. You can survive as long as you'd like by picking big enough constants, but you can't survive forever: once you've picked your constants, you're stuck with them, and once they run out you lose (or you get another chance to pick constants, but only finitely many such chances).

Does this remind you at all of Goodstein sequences?

1.3 Goodstein sequences again

With the ordinals up to ε_0 in hand, we're ready to answer Question 1.2. It turns out that every Goodstein sequence terminates, but they take *exetremely* long to terminate, and the length of the Goodstein sequence for n grows *extremely* quickly as n increases (we might discuss just how quickly another week).

As I'm sure you worked out, the Goodstein sequence for 4 indeed terminates, after only

$$3 \cdot 2^{402{,}653{,}211} - 2 \approx 7 \cdot 10^{121{,}210{,}694}$$

terms. Let's prove this always works.

Theorem 1.4. *Every Goodstein sequence terminates.*

Proof. With each term written in hereditary base b, replace each b with ω. For the Goodstein sequence for 333, this starts

b	Hereditary base b (n_b)	Using ω (o_b)
2	$2^{2^{2+1}} + 2^{2^2+2} + 2^{2+1} + 2^2 + 1$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^\omega + 1$
3	$3^{3^{3+1}} + 3^{3^3+3} + 3^{3+1} + 3^3$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^\omega$
4	$4^{4^{4+1}} + 4^{4^4+4} + 4^{4+1} + 4^3 \cdot 3 + 4^2 \cdot 3 + 4 \cdot 3 + 3$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^3 \cdot 3 + \omega^2 \cdot 3 + \omega \cdot 3 + 3$
5	$5^{5^{5+1}} + 5^{5^5+5} + 5^{5+1} + 5^3 \cdot 3 + 5^2 \cdot 3 + 5 \cdot 3 + 2$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^3 \cdot 3 + \omega^2 \cdot 3 + \omega \cdot 3 + 2$
6	$6^{6^{6+1}} + 6^{6^6+6} + 6^{6+1} + 6^3 \cdot 3 + 6^2 \cdot 3 + 6 \cdot 3 + 1$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^3 \cdot 3 + \omega^2 \cdot 3 + \omega \cdot 3 + 1$
7	$7^{7^{7+1}} + 7^{7^7+7} + 7^{7+1} + 7^3 \cdot 3 + 7^2 \cdot 3 + 7 \cdot 3$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^3 \cdot 3 + \omega^2 \cdot 3 + \omega \cdot 3$
8	$8^{8^{8+1}} + 8^{8^8+8} + 8^{8+1} + 8^3 \cdot 3 + 8^2 \cdot 3 + 8 \cdot 2 + 7$	$\omega^{\omega^{\omega+1}} + \omega^{\omega^\omega+\omega} + \omega^{\omega+1} + \omega^3 \cdot 3 + \omega^2 \cdot 3 + \omega \cdot 2 + 7$

Call the term with base b in the Goodstein sequence n_b, and call the corresponding ordinal number, with ω replacing each b, o_b. Replacing a finite number with ω only makes it bigger, so $o_b \geq n_b$. To show that n_b reaches 0, it's enough to show that o_b reaches 0.

Now o_b is a sequence of ordinal numbers, and I claim it's strictly decreasing. Replacing b with $b+1$ doesn't affect o_b at all, since they're all replaced with ω. Subtracting 1 from n_b makes o_b smaller: it can subtract 1 from o_b, or it can 'break up' a power of ω, replacing it with a sum of finitely many smaller powers of ω. For instance, in the example for 333, at base 4, we replace ω^4 with $\omega^3 \cdot 3 + \omega^2 \cdot 3 + \omega \cdot 3 + 3$, and at base 8, we replace ω with 7. The sum of finitely many smaller powers of ω is less than the original power of ω, so the result is smaller.

So we have a strictly decreasing sequence of ordinal numbers. Since the ordinals are well-ordered, it eventually reaches 0, and then so does n_b. □

1.4 Course overview

We'll survey a handful of topics surrounding the ordinal numbers. I expect each week to be largely independent, except that they all involve the ordinal numbers. Likely topics include

- The fast-growing hierarchy: using ordinals to define really big (but finite) natural numbers
- Ordinal numbers in Chess: can you force checkmate in ω^3 moves?
- Counting to much larger ordinals than ε_0
- Cardinality and uncountable ordinals
- More precise definitions of ordinals and ordinal arithmetic, and proofs of the basic facts we've taken for granted

If there's something you'd like to learn about the ordinals—maybe something I mention in passing or something you've heard about somewhere else—let me know! I can at minimum point you to somewhere you can read about it, and at best it could be the topic of one week's class.

1.5 Exercises

Here are some questions to ponder if you want to think more about ordinal numbers in between classes. HSSP classes don't have homework, so you should only work on these if you want to, and it won't be required for future classes.

Exercise 1.5. Give names to bigger ordinal numbers. We counted up to ε_0, and defined $\varepsilon_1 = \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}}$. What comes next? What's the biggest ordinal you can describe? It might be useful to define notation, e.g. what does ε_x mean?

Exercise 1.6. There's a vaguely similar process to Goodstein sequences, involving cutting heads off of a Hydra which causes more heads to grow back. It's a bit complicated to describe, so see this article[1] for the full rules, but **don't read the proof yet** (the article also links to an interactive applet, but it doesn't seem to work anymore). Prove, using the ideas we discussed for Goodstein sequences, that the Hydra must eventually die.

Exercise 1.7. I claimed the Goodstein sequence for 4 has $3 \cdot 2^{402,653,211} - 2$ terms. Why? Investigate the structure of the sequence to show that this number is correct. How long is the Goodstein sequence for 5?

[1] `http://math.andrej.com/2008/02/02/the-hydra-game/`