Numbers That Do Weird Things, Week 2

HSSP Summer 2016

2016-07-17

Any problem/subproblem that starts with (T) requires trigonometry. Have fun!

1. Quaternions and Computer Graphics (Problem from nrich.maths.org)

   (a) Consider the quaternion $q = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + 0j + 0k$

      i. Show that the multiplicative inverse of q is given by $q^{-1} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

      ii. Show that for all scalar multiples $x = ti$ of i, $qx = xq$ and hence $qxq^{-1} = x$. This proves that the map $F(x) = qxq^{-1}$ fixes every point on the x axis.

      iii. What happens to points on the y axis under the mapping F? To answer this work out $F(j)$. Also compute $F(k)$.

   (b) (T) Consider the quaternion $q = \cos + (\sin)k$.

      i. Show that $\cos\theta(sin\theta)k$ is the multiplicative inverse of q.

      ii. Show that $qkq^{-1} = k$.

      iii. Show that $qvq^{-1} = r(\cos(2\theta + \phi)i + sin(2\theta + \phi)j)$ where $v = (r(cos\theta)i + (sin\theta)j + 0k)$ and hence that the map $G(v) = qvq^{-1}$ is a rotation about the z axis by an angle 2θ.

   So quaternions can be used to represent transformations in 3D, and they only have 4 numbers to store. On the other hand, a 3D rotation matrix has 9 numbers. Just with this you can see how quaternions are useful in computer graphics. If you'd like to read more, check out this article: https://plus.maths.org/content/os/issue42/features/lasenby/index

2. Cayley-Dickson on a Finite Field

   Apply the Cayley-Dickson construction to $\mathbb{Z}/3\mathbb{Z}$: the integers mod 3. Write a multiplication table.

3. Associativity-Like Properties

   Show that an algebra that is alternative is also power associative (if you don't rememberr the meanings of these terms, look them up!).

4. Different Involutions and Cayley Dickson

(a) Consider the involution $f(a+bi) = -a-bi$, and call the complex numbers with this involution $\mathbb{C}^*$. What happens when we apply Cayley-Dickson to $\mathbb{C}^*$ (ignoring the way defined the involution in Cayley-Dickson originally)?

(b) Suppose we use the involution $f(x) = \frac{1}{x}$ on $\mathbb{R}$. What can we say about the norm? Prove it!

(c) Suppose we use the involution $f(x) = \frac{1}{x}$ on $\mathbb{R}$ to build some structure $\mathbb{C}^*$ with Cayley-Dickson. What can we say about the norm? What happens to multiplication? Is $\mathbb{C}^*$ even an algebra? Prove your claims!

5. Involutions

   (a) Prove that if f is an involution on a finite set with an odd number of elements then f has an odd number of fixed points.

   (b) Prove that if f is an involution on a finite set with an even number of elements then f has an even number of fixed points.

6. Matrices

   (a) Prove or disprove: There exists a matrix representation of the octonions.

   (b) Come up with one or more involutions on the set of 2x2 matrices.