Week 2: Chess and Trees

Dylan Hendrickson | *MIT Educational Studies Program*

2.1 Checkmate in n

A typical Chess puzzle presents a board state, and asks whether White can checkmate within n moves, usually for some small value of n. Roughly, this means there is a strategy White can follow such that no matter what Black does, White will win within n of White's turns. We're going to look for checkmates in n moves when n is an infinite ordinal, so let's be precise about exactly that means, in a way that lets us extend the definition from finite numbers to ordinals.

Unlike most Chess puzzles, it will be more convenient to think about how many turns White needs to make when it's Black's turn.

Definition 2.1. Given Chess position with Black to play, White has *mate in n* if no matter what move Black makes, White can move such that the resulting position is a mate in k for some $k < n$.

A *mate in* 0 is a position in which Black is already checkmated, so White has already won. You can think of this as a special case, or as a vacuous instance of the general definition.

This definition is recursive: it defines 'mate in 1' in terms of 'mate in 0', defines 'mate in 2' in terms of both 'mate in 1' and 'mate in 0,' and so on.

- A mate in 0 is when White has already won.
- A mate in 1 is when White can achieve a mate in 0 (and win) by their next turn.
- A mate in 2 is when White can achieve ether a mate in 0 or a mate in 1 by their next turn.
- etc.

Exactly how long White takes to win can depend on Black's play. For instance, if White has mate in 2 and it's Black's turn, Black might have a move that lets white win that turn. But Black might also have a move which delays White winning by an extra turn.

This definition counts mates which don't require all n moves—for instance any mate in 1 is also a mate in 3. To talk about the number of turns White needs to win, let's define another word.

Definition 2.2. The *value* of a Chess position with Black to play is the smallest n such that White has a mate in n, if such an n exists.

Unpacking these definitions gives us a way to compute the value of a position.

Lemma 2.3. *The value of a position P is the smallest number which is greater than the value of each position resulting from White's optimal response to one of Black's possible moves from P.*

By 'optimal,' we mean that White always prefers to move to positions where they have mate sooner, i.e. positions with smaller value. This lemma is probably either hard to understand or obviously true, depending on how well you understand the definitions. It's worded carefully, and maybe confusingly, so that it will keep working with infinite ordinal numbers. Let's prove it carefully.

Proof. Let n be the number described in the lemma statement. To show that P has value n we need to show that White has mate in n, and also that White doesn't have mate any faster, so n is the smallest such number. We're going to use induction, meaning to prove that the lemma holds for P we will assume that it holds for the positions one turn (for each player) after P.

White has mate in n: After Black makes some move from P, White plays their optimal response, resulting in a position with value k. By definition, n is greater than k, and now White has mate in k. Since this works for any move Black could make, P meets the definition of mate in n.

White doesn't have mate faster than n: Suppose White has mate in k. That means that for each of Black's possible moves i, White has a move to a position Q where they have mate in ℓ with $\ell < k$. So White's optimal response, which must be at least as good as Q, also has mate in less than k. By our inductive hypothesis, the value of White's optimal response is less than k. So k is greater than the value of each of the positions resulting from White's optimal responses, but n was defined as the smallest such number, so $k \geq n$. □

Exercise 2.4. In Definition 2.2, how do we know that if there's any n for which White has a mate in n, then there's a smallest such n?

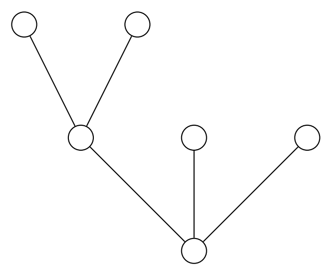
Exercise 2.5. What does mate in ω mean? What about $\omega + 1$?

2.2 Infinite Trees

One useful way to represent games is as a tree of all possible futures.

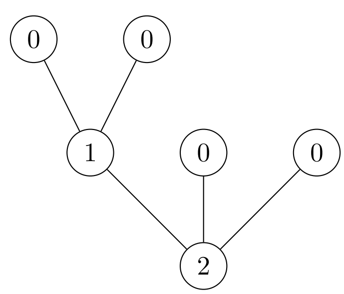
Definition 2.6. A *tree* has a *root* which has some number of *children*, each of which is a root of another tree.

Here's a simple tree:



The root is at the bottom, and has three children. Its first child has two children, each of which has no children, and the other two children also have no children. Note that each node in the tree is also a root of its own smaller tree, or *subtree*. We don't always distinguish between a (sub)tree and its root. A node with no children is called a *leaf*.

We label each node with the 'height' of its subtree, in the intuitive way:

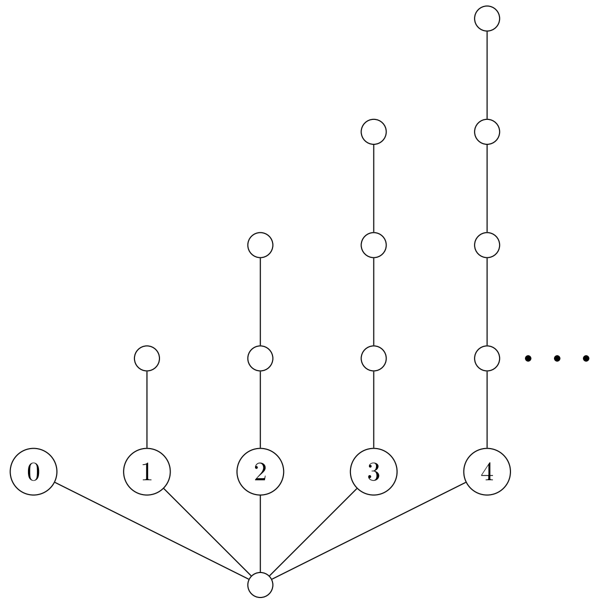


We'll want to extend this to infinite ordinal numbers, so let's define height more carefully.

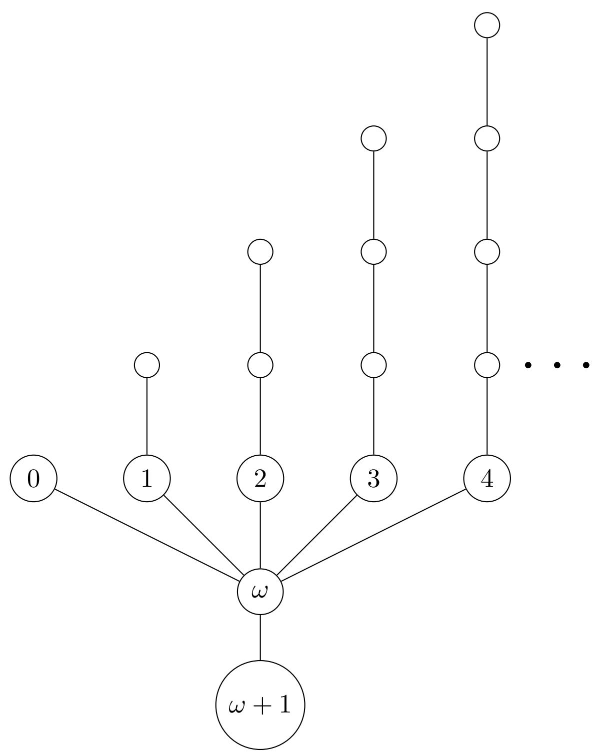
Definition 2.7. The *height* of a tree is the smallest number which is greater than the heights of each of its children.

Does this sound anything like Chess yet?

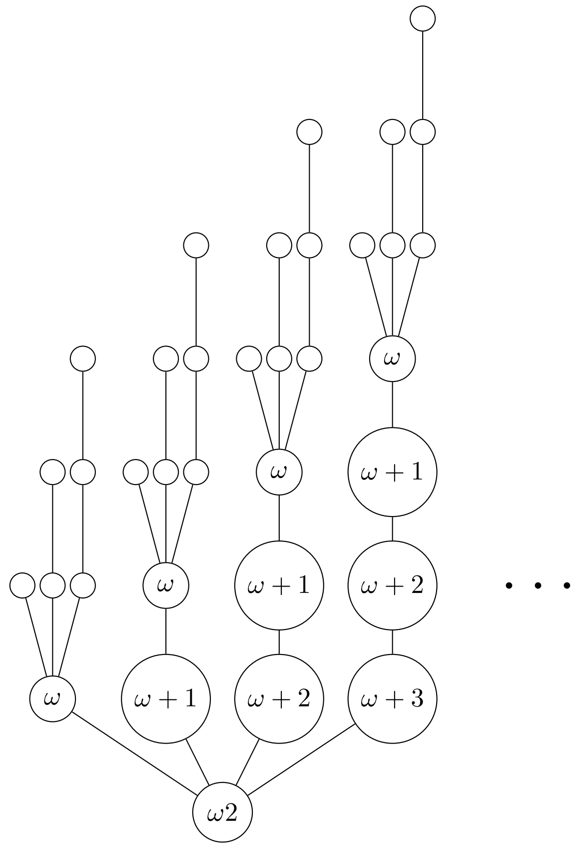
Let's get some ordinal numbers involved, and see why the definition is phrased awkwardly. Consider this tree:



This tree has infinitely many children, with heights 0, 1, 2, and so on. What's its height? The definition says its the smallest number greater than all of 0, 1, 2, and so on. We gave this number a name: ω. We can keep going, building trees with larger ordinal numbers as heights.



We'll draw less, and omit some dots, to fit bigger trees.



You can continue like this to build trees whose heights are whatever ordinals you want.

Exercise 2.8. If a tree's children all have ordinal numbers heights, is the height of the whole tree well-defined and an ordinal number?

2.3 Climbing Trees

Let's consider a game called *Tree Climbing*, which involves climbing trees. There are two players, White and Black. We start at the root of a tree. Black moves by choosing a child of the current position and moving there. White's only move is to nod in approval. Black loses when we reach a leaf, since they would have nowhere to go.

Exercise 2.9. Prove that if a tree's height is any ordinal number, then Black must eventually lose.

We can analyze Tree Climbing positions just like Chess positions. Using the same definitions for 'mate in n' and 'value,' Lemma 2.3 says that the value of a Tree Climbing position is exactly its height.

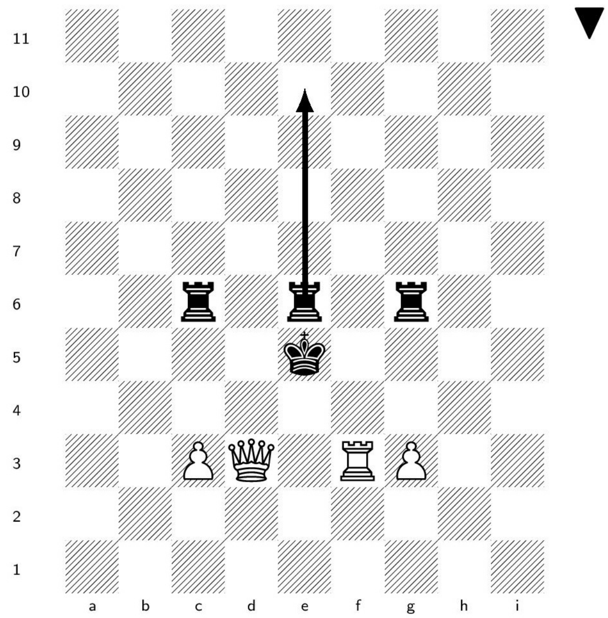
2.4 Chess Positions with Ordinal Value

After all that abstract nonsense, let's see some actual Chess positions whose values are infinite ordinals. Importantly, for this to happen there need to be infinitely many positions reachable with a single move—to build a tree of height ω, we need a node with infinitely many children. So these Chess positions are on an infinite board, and we get this infinite fanout from the fact that pieces like rooks can move any number of spaces. The distance Black moves a rook determines the value of the position afterwards, just like choosing one of the finite-height children of a node with height ω. Most of the positions also use infinitely many pieces, continuing the pattern in the finite pictures.

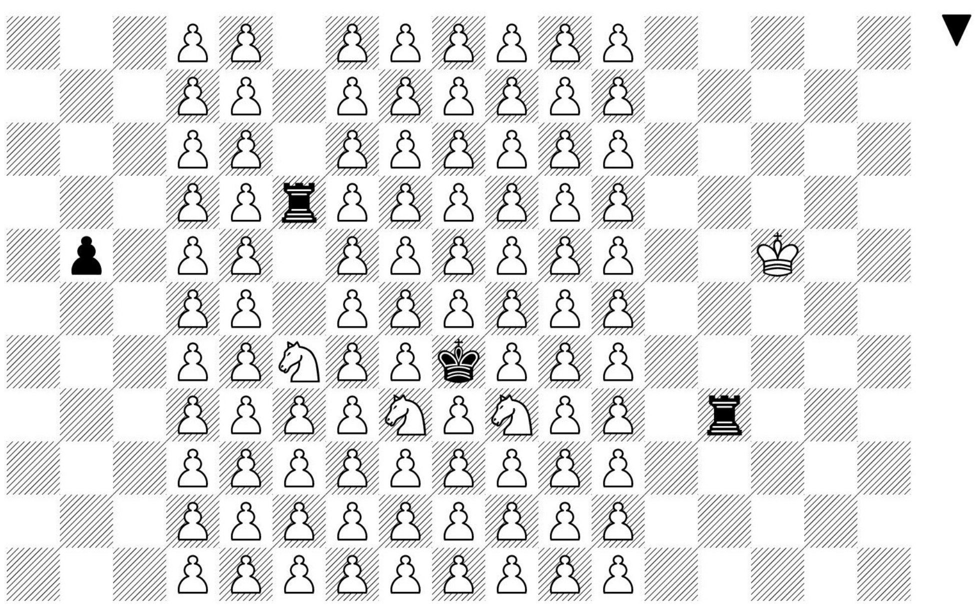
The actual Chess positions are from Joel David Hamkins' blog, and you should go there for full explanations, though pictures of them are also included below. The largest value shown is ω^4; nobody knows whether there are Chess positions with value larger than ω^4. Maybe you could find one!

The first four positions below are from `http://jdh.hamkins.org/game-values-in-infinite-chess/`, and the last is from `http://jdh.hamkins.org/a-position-in-infinite-chess-with-game-value-omega-to-the-4/`.

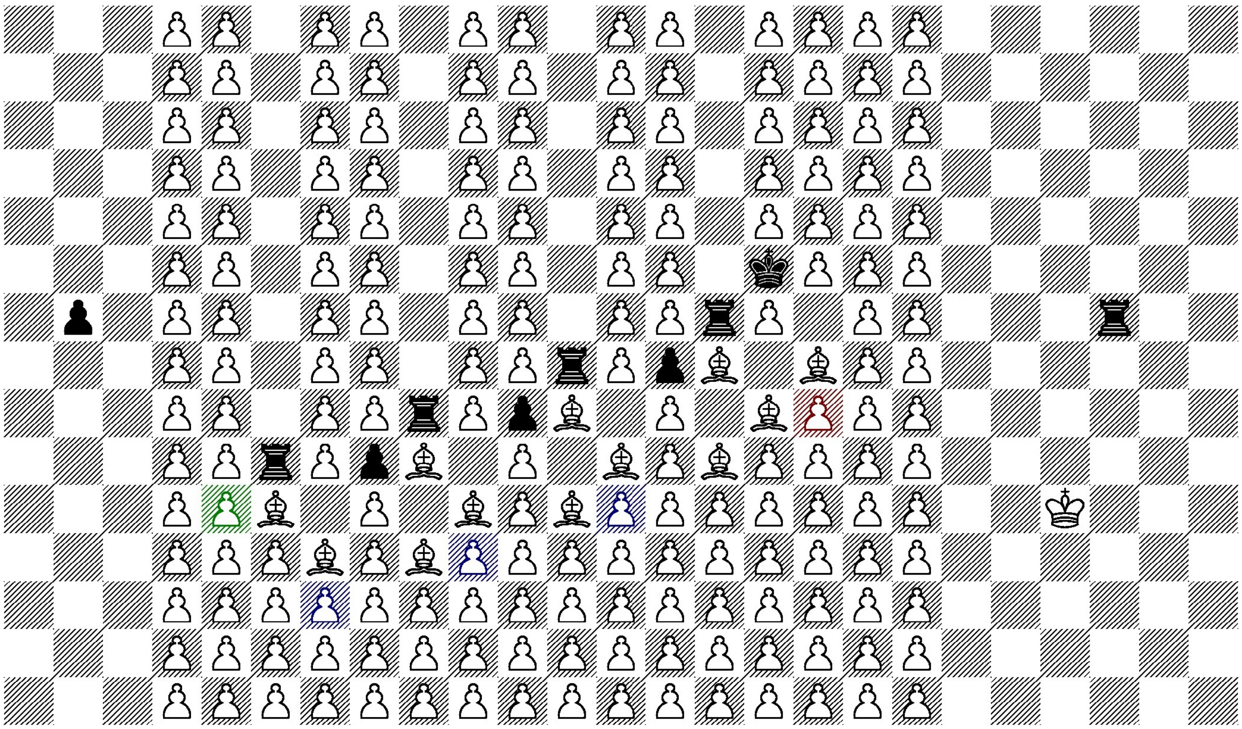
Value ω:



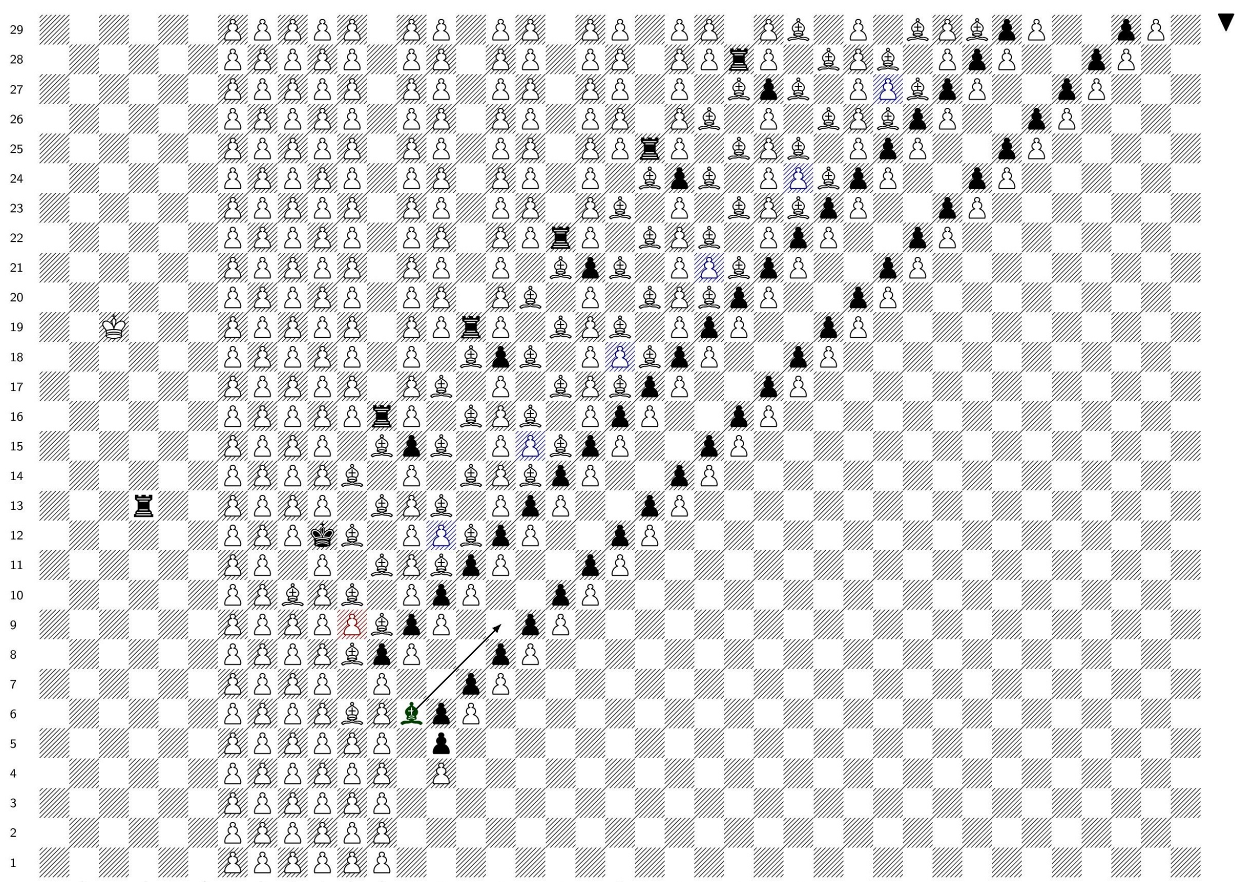
Value ω^2:



Value $\omega^2 4$:



Value ω^3:



Value ω^4: