Week 3:

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3.1 Sizes of sets

If you don't know how to count, how can you tell whether you have the same number of fingers on each hand? The easiest way is to pair up the fingers: if you can match each finger on your left hand with a finger on your right hand such that each finger is used exactly once, you must have the same number on each hand. This inspires some definitions:

Definition 3.1. A *bijection* between two sets A and B is a way to match elements of A with elements of B such that each element is used exactly once. If a bijection exists, we say that A and B have the same *cardinality*, and write $A \cong B$.

Cardinality is the simplest notion of the "size" of a set. It behaves nicely in ways you'd expect:

Exercise 3.2. Prove, for any sets A, B, and C, that

1. $A \cong A$.
2. If $A \cong B$, then $B \cong A$.
3. If $A \cong B$ and $B \cong C$, then $A \cong C$.

Proving these involves describing a matching in terms of other matchings. Taken together, they say that $\cong$ is something called an 'equivalence relation,' which means it's a reasonable notion when things (in this case, sizes of sets) are the same.

For finite sets, cardinality behaves exactly as one would hope:

Exercise 3.3. Suppose A has m elements and B has n elements, where m and n are finite numbers. Show that if $m = n$, then there's a bijection between A and B, and if $m \neq n$, then there isn't a bijection. (What does it mean when we say A "has m elements"?)

For infinite sets, it gets a bit weirder.

3.2 Infinite sets

To help name infinite sets, let's think of an ordinal number as describing the set of ordinals less than it. For convenience, we can say that an ordinal number *is* the set of ordinals less than it. For instance (don't worry if some of this notation doesn't make sense):

$$\begin{aligned}
0 &= \emptyset = \{\} \\
3 &= \{0, 1, 2\} \\
\omega &= \{0, 1, 2, 3, \cdots\} \\
\omega + 1 &= \{0, 1, 2, 3, \cdots, \omega\} = \omega \cup \{\omega\} \\
\omega 2 &= \{0, 1, 2, \cdots, \omega, \omega + 1, \omega + 2, \cdots\} \\
\omega^2 &= \{0, 1, 2, \cdots, \omega, \omega + 1, \omega + 2, \cdots, \omega 2, \omega 2 + 1, \omega 2 + 2, \cdots, \cdots\} = \{\omega a + b \mid a, b \in \omega\}
\end{aligned}$$

Let's think about the cardinality of infinite ordinals. As ordinal numbers, $\omega < \omega + 1$, but is $\omega \cong \omega + 1$?

Exercise 3.4. Answer this question. That is, describe a bijection between the ordinals less than ω and the ordinals less than $\omega+1$, or argue that there isn't one.

So we can add one element to an infinite set without changing its cardinality (in fact, that's a reasonable definition of "infinite"). We could do this repeatedly, so $\omega \cong \omega+1 \cong \omega+2 \cong \cdots$. What about $\omega 2$?

Exercise 3.5. Describe a bijection between ω and $\omega 2$.

It seems lots of sets have the same cardinality as ω. We'll call such a set *countable*, because you can 'count' its elements with $0, 1, 2, \cdots$ (finite sets are also usually considered countable). Since we're talking about ordinal numbers as sets, an ordinal number is countable if there are countably many ordinals smaller than it. It's hopefully clear that we can use the same ideas to show that ωn is countable for any finite n. What about ω^2?

You can think of the ordinals less than ω^2 in a 2D array:

$$\begin{array}{ccccc}
0 & 1 & 2 & 3 & \cdots \\
\omega & \omega+1 & \omega+2 & \omega+3 & \cdots \\
\omega 2 & \omega 2+1 & \omega 2+2 & \omega 2+3 & \cdots \\
\omega 3 & \omega 3+1 & \omega 3+2 & \omega 3+3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

To show that there are countably many, we just need to label each one with a natural number $0, 1, 2, \cdots$. We can do this starting in the top left corner, and listing each diagonal row in order, like this:

$$\begin{array}{ccccc}
0 & 1 & 3 & 6 & \cdots \\
2 & 4 & 7 & 11 & \cdots \\
5 & 8 & 12 & 17 & \cdots \\
9 & 13 & 18 & 24 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The next interesting case is ω^3. You can think of the ordinals less than ω^3 in a 3D array, and find an order to list them. Then for ω^4, just put them in a 4D array, and so on.

But it's simpler if we use the fact thot ω^2 is countable to prove that ω^3 is countable. Since ω^2 is countable, we can list the ordinals less than ω^2 in a single (1D) row of the table. Then ω^3 contains ω copies of ω^2, so we can list the elements of ω^3 in a 2D table, and count them the same way as we did for ω^2. Repeating this shows that ω^n is countable for any finite n.

Instead of thinking about this increasing-diagonal-order bijection every time, let's turn it into a lemma:

Lemma 3.6. *Suppose sets $A_0, A_1, A_2, \cdots$ are all countable. Then the union of all of them $\bigcup_{i\in\omega} A_i$ is also countable. More succinctly, the union of countably many countable sets is countable.*

This is the key step in proving that ω^3 is countable: we split it apart into ω copies of ω^2, which is countably many countable sets.

Proof. List the elements of A_i in the ith row of a table, using the fact that A_i is countable. We can count the entire table using the order described above. $\square$

If there are duplicate elements in the A_i's, the lemma is still true, since duplicates can only make the union smaller that it would be without duplicates. If you'd like, you can skip over any duplicates when listing the elements in the diagonal order.

Exercise 3.7. Is ω^ω countable?