e and the Complex Numbers: Supplement 1

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1 Summary of Days 1 and 2

1.1 e

There are numerous ways to represent the number e; in addition to its decimal expansion, it can be written as a limit, an infinite sum, or one of several continued fractions.

- $2.718281828459045\cdots$
- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
- $\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$
- The continued fractions $[2,1,2,1,1,4,1,1,6,1,1,8,1,\cdots]$ or $[1,0,1,1,2,1,1,4,1,1,6,1,1,8,1,\cdots]$

It's a pretty special number, and the function e^x and its inverse $\ln x$ have some interesting properties:

- $\frac{d}{dx}e^x = e^x$
- $e^{i\theta} = \cos\theta + i\sin\theta$, which reduces to $e^{i\pi}+1=0$ when $\theta=\pi$.
- $\frac{d}{dx}\ln x = \frac{1}{x}$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ when $0 < (1+x) \le 2$.

This last property is what made the "natural logarithm" easy to calculate compared to logarithms in other bases. This was helpful for John Napier in developing logarithms as a multiplication shortcut.

Napier's also took advantage of the identity $\log(xy) = \log x + \log y$. Using this rule, one can quickly approximate any logarithm from a table of precomputed values between 1 and 10.

The same identity can also be written as $a^{x+y} = a^x a^y$. The similarity of this to the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$ suggests a connection between exponential functions and trigonometric functions. We use complex numbers to investigate this connection.

1.2 Complex Numbers

The motivation for having complex numbers at all comes from trying to solve polynomials. For instance, many quadratics such as $x^2 = -1$ are unsolvable unless complex solutions are permitted. A stronger motivation comes from cubics; Cardano's method for solving cubic polynomials gives answers of the form $\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}$. Cubics with three real solutions produced negative values of b, highlighting the need to invent a number i whose square is equal to -1.

Since i can't really be said to lie between real numbers, we can visualize it and all its multiples on an axis perpendicular to the real line. Taking all possible sums of real and imaginary numbers gives the complex plane, a space in which multiplying by i is the same as a 90-degree rotation. We'll write "$\mathbb{C}$" for the set of all complex numbers.

1.3 Some Trigonometry

The function $\operatorname{cis}\theta = \cos\theta + i\sin\theta$ is special:

- It behaves somewhat like an exponential function, since it follows the identity $\operatorname{cis}(x+y) = \operatorname{cis} x \operatorname{cis} y$
- It is one of only three linear combinations of sine and cosine that have the above property. (The other two are $\cos\theta - i\sin\theta$ and boring old zero.)
- $(\operatorname{cis} x)^n = \operatorname{cis}(nx)$ (de Moivre's formula)

1.4 Graphs of Complex Functions

We'd like to graph functions from complex numbers to complex numbers, but we don't have enough dimensions for all 4 axes. Instead we assign a color to every point in the complex plane: a color from the rainbow denotes the angle that separates a complex number from the positive real axis, and brightness denotes its distance from the origin. To graph a function $f(z)$, we paint every point z in $\mathbb{C}$ the color of $f(z)$.

2 Euler's Derivation of the Natural Logarithm's Taylor Series

This derivation of the Taylor series for $\ln(1+x)$ is attributed to Euler. It isn't quite what we could call rigorous proof, but it's an interesting way of looking at the natural logarithm.

Split $\ln(1+x)$ into the product ωn where ω is really small and n is positive and really big. Then:

$$\begin{aligned}\omega &= \frac{1}{n}\ln(1+x) \\ &= \ln(1+x)^{\frac{1}{n}} \\ &= \ln\left[1+\left((1+x)^{\frac{1}{n}}-1\right)\right]\end{aligned}$$

From $\frac{d}{dx}e^x = e^x$ we obtain the linear approximation $e^x \approx 1+x$ for x near zero, which can be rewritten as $\ln(1+x) \approx x$. Then:

$$\begin{aligned}\omega &= \ln\left[1+\left((1+x)^{\frac{1}{n}}-1\right)\right] \\ &\approx (1+x)^{\frac{1}{n}}-1\end{aligned}$$

Newton's generalization of the binomial theorem can be used to simplify this:

$$\begin{aligned}(1+x)^{\frac{1}{n}}-1 &= \left[1+\frac{1}{n}x+\frac{\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)}{2!}x^2+\cdots\right]-1 \\ &= \frac{1}{n}x+\frac{\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)}{2!}x^2+\frac{\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{3!}x^3+\cdots\end{aligned}$$

Recalling that this expression is supposed to approximate ω and that we let $\ln(1+x) = \omega n$, multiply the expression by n to obtain an expression for $ln(1+x)$:

$$\ln(1+x) \approx x+\frac{\left(\frac{1}{n}-1\right)}{2!}x^2+\frac{\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{3!}x^3+\cdots$$

Since n is large, $1/n$ is almost zero, which allows us to simplify this into a familiar sum.

$$\begin{aligned} \ln(1+x) &\approx x + \frac{(-1)}{2!}x^2 + \frac{(-1)(-2)}{3!}x^3 + \frac{(-1)(-2)(-3)}{4!}x^4 + \cdots \\ &\approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \end{aligned}$$

The generalized binomial theorem was important here, and there's actually a lot hiding behind it. It was known by Euler's time, but people had difficulty establishing a solid proof of it. To illustrate, one might try to derive it by finding the Taylor expansion of $(1+x)^y$ for a real number y. That, unfortunately, requires a rule for finding $\frac{d}{dx}x^y$ for arbitrary reals y, and the usual derivation of such a rule uses the generalized binomial theorem!